1 Basic Functions and Equations

Straight lines, linear equations

1. (a) Find the equation of the line through the points $A(1,-3)$ and $B(5,3)$.

   (b) Also find the equations of the lines (i) parallel to AB and (ii) perpendicular to AB through the point $C(-1,4)$.

2. (a) Use your calculator to sketch the graphs of $y=-2x-2$ and $y=\frac{1}{3}x+4$, and from the display estimate the coordinates of their point of intersection.

   (b) What are the gradients and y-intercepts of the lines with equations (i) $4x-2y=7$ and (ii) $2x+5y-3=0$?

   (c) Find the x-intercepts of the lines in parts (a) and (b) of this problem.

3. Find the equation of the line through $P(-1,5)$ perpendicular to the line $y=\frac{1}{2}x-2$, and the point F of intersection of the two lines. What is the length of FP ? What then is the distance of the point P from the first line?

4. A triangle has vertices $P(-3,-2), Q(5,-6), R(5,10)$.

   (a) Find the equations of the three altitudes of the triangle. (An altitude or height is a line through a vertex of a triangle perpendicular to the opposite side.)

   (b) Find the point I of intersection of the altitudes through Q and R.

   (c) Show that the altitude through P also passes through I. (The three altitudes of the triangle are therefore said to be 'concurrent'.)

   (d) Find the length of that altitude through P and the length of the side QR of the triangle, and hence calculate the area of the triangle PQR.

5. A triangle has vertices $A(-3,5), B(1,-2)$ and $C(8,6)$.

   (a) Find the coordinates of the midpoints K, L and M of the sides BC, AC and AB.

   (b) Hence find the equations of the three medians AK, BL and CM of the triangle. (A median is a line through a vertex of a triangle and the midpoint of the opposite side.)

   (c) Determine the point G of intersection of the medians AK and BL.

   (d) Show that the third median CM also passes through G. (So the three medians of the triangle are concurrent.)

   (e) What is the ratio of the lenghts of AG and GK? How do you think you can generalise this?

   (f) The point G is called the centroid, and it is the centre of gravity of the triangle. Can you suggest a way in which the coordinates of G could have been calculated from the coordinates of the three vertices?

6. Solve the following simultaneous equations for the pairs of values of a and b given below.

$$\begin{array}{rcrcrcr} 2x & - & y & + & 2z & = & -1 \\ 3x & + & a\,y & - & z & = & 1 \\ x & - & 4y & + & 5z & = & b \end{array}$$

(a) $a = -2, \quad b = -15,$ (b) $a = 2, \quad b = -3,$ (c) $a = 2, \quad b = 2.$

Quadratic equations and parabolas

7. By substituting the coordinates of the points and solving the simultaneous equations, find the values of a, b and c such that the graph of the function $y = ax^2 + bx + c$ passes through the points $A(-2, 13), B(1, -2)$ and $C(3, 8)$.

8. Solve the following quadratic equations, either by factorising or by completing the square.

(a) $x^2 + x - 6 = 0$ (b) $x^2 - 5x - 6 = 0$ (c) $x^2 + 3x - 40 = 0$
(d) $3x^2 + 5x - 2 = 0$ (e) $4x^2 + 4x - 3 = 0$ (f) $4x^2 + 11x = 3.$

9. Let $f(x) = x^2$. Using your calculator or otherwise sketch the following graphs, and in each case write down the coordinates of the vertex. Try to generalise.

(a) $y = f(x)$ (b) $y - 2 = f(x)$ (c) $y + 1 = f(x)$
(d) $y = f(x - 3)$ (e) $y = f(x + 2)$ (f) $y + 3 = f(x - 1)$
(g) $y = -f(x)$ (h) $y - 1 = -f(x)$ (i) $y = -f(x + 4)$

10. Rewrite each of the following functions in the form $y = m(x - p)^2 + q$, and hence write down the coordinates of the vertex in each case. (Hint: multiply out, or expand, the given general form and compare the quadratic, linear and constant terms; then use what you learnt in 9.)

(a) $y = x^2 - 6x + 11$ (b) $y = x^2 + 3x + 2$ (c) $y = 2x^2 - 4x - 3$

11. Write down the quadratic equations with integer coefficients whose roots (i.e. solutions) are

(a) $x = -3, 2$ (b) $x = \frac{1}{2}, 6$ (c) $x = 2 \pm \sqrt{3}$ (d) $x = 3$

12. (a) Given the parabola with equation $y = x^2 - 3x - 4$, find

i. its intersections with the line with equation $y = -2x + 2$,

ii. the value of c such that the parabola meets the line $y = -2x + c$ in only one point. (This kind of line is called a 'tangent' of the parabola.)

In one diagram, sketch the parabola and the two lines.

(b) Given the parabola with equation $y = \frac{1}{2}x^2 - 2x + 5$, find the gradients of the two tangents through the origin. Again, sketch the parabola and the two tangents.

13. Given the parabolas with equations $y = 2x^2 - 3x - 5$ and $y = x^2 - x + 3$,

(a) find their two points of intersection and the equation of the line through those points,

(b) combine the equations of the two curves to eliminate their quadratic terms.

2 Trigonometry

Trigonometric ratios in right-angled triangles

1. Draw a right-angled triangle with another angle equal to $40°$ as accurately as you can, and measure the lengths of the three sides. From these lengths calculate the approximate values of the six trigonometric ratios for $40°$, i.e. $\sin 40°$, $\cos 40°$, etc.
   Compare these approximate values with the values obtained using the calculator. What is the percentage error of your approximate value of $\cos 40°$?

2. Without finding the angles θ, find the values of the other five trigonometric ratios, given that
   (a) $\sin\theta = 0.6$ (b) $\sec\theta = 2$ (c) $\tan\theta = \dfrac{12}{5}$ (d) $\cos\theta = 0.4$.

3. If $\cos 15° = a$, find the following in terms of a:
   (a) $\tan 15°$ (b) $\sin 75°$ (c) $\sec 75°$ (d) $\tan 75°$.

4. Without using a calculator, find the angle θ, where $0° \le \theta \le 90°$ such that
   $\sin 60° \cos 30° - \cos 60° \sin 30° = \sin\theta$.

... and in the unit circle

As far as possible, do these questions without a calculator and give your answers as precise numbers.

5. By finding the associate acute angles and determining the quadrant in which the angles are, find the following values of trigonometric functions:
   (a) $\sin 150°$ (b) $\cos 330°$ (c) $\tan 120°$ (d) $\cos 225°$
   (e) $\cot 300°$ (f) $\sec 135°$ (g) $\tan(-45°)$ (h) $\sin(-240°)$
   (i) $\cos 1200°$.

6. Find the values of the other five trigonometric functions of the angle θ if
   (a) $\cos\theta = -0.6$ and θ is in quadrant III
   (b) $\csc\theta = 3$ and θ is in quadrant II .

7. Find the values of the angle θ, where $0° \le \theta \le 360°$, such that
   (a) $\cos\theta = \dfrac{1}{2}$ (b) $\tan\theta = -\sqrt{3}$ (c) $\sin\theta = -\dfrac{\sqrt{3}}{2}$ (d) $\tan\theta = 1$
   (e) $\sec\theta = -\sqrt{2}$ (f) $\cot\theta = \sqrt{3}$ (g) $\cos\theta = -0.1$ (h) $\csc\theta = 4$
   (i) $\cos\theta = 0$.

8. For what angle θ, $0° \le \theta \le 360°$, is
   (a) $\sin\theta = \dfrac{1}{2}$ and $\cos\theta < 0$ (b) $\sec\theta = -\sqrt{2}$ and $\tan\theta > 0$
   (c) $\tan\theta = 1.5$ and $\sin\theta < 0$?

9. Find all the solutions from $0°$ to $360°$ of the following trigonometric equations:

(a) $\tan 2x = \sqrt{3}$ (b) $\sin(x + 30°) = -\frac{\sqrt{2}}{2}$ (c) $\cos\frac{x}{3} = 0.6$

(d) $\sin(2x - 45°) = 0.5$ (e) $\cot^2 x = 3$ (f) $\sec(x + 1°) = -3$.

10. Find the general solutions of the following equations, and try to simplify them:

(a) $\cot(x - 15°) = -1$ (b) $\sin(2x + 30°) = \frac{1}{3}$ (c) $\sin(x + 90°) = \frac{\sqrt{2}}{2}$

(d) $\cos 2x = 0.5$ (e) $\tan x = \sqrt{3}$.

Circular measure; Sectors, segments

11. (a) Change the following angles into degrees:

$\frac{\pi}{3}$, $\frac{4\pi}{3}$, -3π, $\frac{\pi}{5}$, $\frac{13\pi}{12}$, 3.14, 1, $\frac{7\pi}{6}$, 0.1, $\frac{\pi}{4}$.

(b) Change the following angles into radians:

$135°$, $30°$, $150°$, $-300°$, $67.5°$, $780°$, $121.6°$, $270°$.

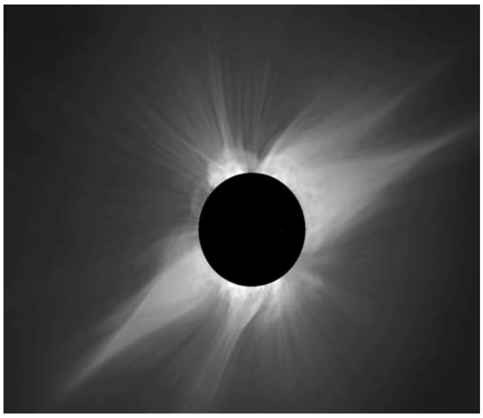
(c) Solve the following equations, giving your answers in radians:

i. $\tan(x + \frac{\pi}{6}) = -1$, $0 \leq x \leq 2\pi$

ii. $\cos\frac{x}{2} = 0.2$, the general solution.

12. (a) Looking at the moon from the earth, it measures $31' \, 7''$ across. And the average distance of the moon from the earth is 384,400 km. By approximating the diameter of the moon by a length of arc, estimate the diameter of the moon. (Note that when measuring angles, one degree is divided into 60 minutes, and one minute further into 60 seconds, so $1° = 60'$ and $1' = 60''$.)

(b) The sun has a diameter of 1.39×10^6 km, and its distance from the earth is 1.5×10^8 km. What angle does the sun therefore subtend at the earth?



1991 Solar Eclipse, by Steve Albers

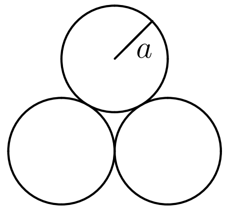
13. In a circle of radius r, the sector with an angle θ at the centre has length of arc l and area A. For each of the sectors given below, find the missing pieces of information.

(a) $r = 5$ cm, $\theta = 120°$ (b) $r = 72$ m, $l = 48$ m

(c) $r = 12$ cm, $A = 100$ cm^2 (d) $\theta = 240°$, $l = 48$ cm

(e) $\theta = \frac{3\pi}{4}$, $A = 20$ cm^2 (f) $l = 20$ cm, $A = 90$ cm^2.

14. (a) Find the area enclosed by three circular discs, all of radius a, lying on a flat table in contact with each other.



(b) Three circular discs with radii 10 cm, 20 cm and 30 cm are lying on a flat table in contact with each other. Find the lengths of the sides and the angles in the triangle whose vertices are the centres of the discs. Hence find the area enclosed between the three discs.

15. A and B are points on a circle of radius 12 cm. Find the area of the smaller segment cut off by the chord AB, if

(a) the chord subtends an angle of 30° at the centre; give the answer in the form $p + q\pi$, where $p, q \in \mathbb{Z}$;

(b) the length of the chord is 12 cm; give your answer in the form $p\sqrt{3} + q\pi$, where again $p, q \in \mathbb{Z}$;

(c) the length of the chord is 18 cm.

16. In a UWC symbol the two circles are of diameter 33 mm and they overlap so that the symbol measures 60 mm across.
How long is the common chord of the two circles, and what is the area of overlap? How long is the perimeter of the whole shape?

17. Find the radius r of the circle in which a chord which subtends an angle of $75°$ at the centre cuts off a segment of area 40 cm^2.

18. P and Q are two points on the circumference of a circle of radius a, and the chord PQ subtends an angle θ at the centre C of the circle. Obtain an equation for θ such that the smaller segment cut off by PQ has the same area as the triangle PQC. Use your calculator to find the solution of that equation, accurate to 1°; (consider carefully though which unit to use when you find θ ! Note that there is in general no method for solving an equation of that form.)

Some trigonometric equations

19. Find **all** the solutions of the following equations, where $0 \le x \le 2\pi$, or $0° \le x \le 360°$. Give your answers to 3 s.f., in unit 1 (radians) for the equations on the left, and in degrees for the equations on the right.

(a)	$4\sin^2 x - 3 = 0$	(b)	$2\sin x\cos x + \sqrt{3}\cos x = 0$
(c)	$2\cos^2 x - 3\cos x + 1 = 0$	(d)	$2\cos^2 x + \sin x - 1 = 0$
(e)	$6\sin^2 x + \cos x - 4 = 0$	(f)	$2\cos^2 x + 5\sin x - 4 = 0$
(g)	$\sin^2 x - 2\cos x - 1 = 0$	(h)	$\sec^2 x - 2\tan x - 1 = 0$
(i)	$\tan^2 x - 3\sec x + 3 = 0$	(j)	$4\sin x - 3\cos x = 0.$
(k)	$\tan x - 2\cot x = 1$	(l)	$2\tan x\sec x - 3 = 0.$

20. Use your calculator to find **a** solution of $4\sin x - 3\cos x = 1$.

Inverse trigonometric functions

21. Show that $\arcsin x + \arccos x = \frac{\pi}{2}$
    (a) by sketching the graphs of $y = \arcsin x$ and $y = -\arccos x + \frac{\pi}{2}$,
    (b) by applying the sine-function on the left-hand side.

22. Simplify
    (a) $\sin(\arcsin x)$,
    (b) $\cos(\arcsin x)$,
    (c) $\sec(\arccos x)$,
    (d) $\tan(\arccos x)$,
    (e) $\cos(\arctan x)$,
    (f) $\cos(2\arctan x)$,
    (g) $\arctan x + \arctan \frac{1-x}{1+x}$.

23. Show that
    (a) $\arccos \frac{3}{5} + \arccos \frac{4}{5} = \frac{\pi}{2}$,
    (b) $\arctan \frac{1}{3} + \arctan \frac{1}{2} = \frac{\pi}{4}$,
    (c) $2\arctan \frac{1}{2} = \arcsin \frac{4}{5}$,
    (d) $\arctan 3 + 2\arctan 2 = \text{arccot}\, 3$.

24. Solve the following equations:
    (a) $6\arccos 2x = \pi$,
    (b) $\arctan(1+x) + \arctan(1-x) = \arctan 2$,
    (c) $\arctan 2 = \arctan 4 - \arctan x$,
    (d) $2\arcsin x = \arccos x$.

3 Vectors

Vector Arithmetic

1. If $ABCDEF$ is a regular hexagon, and $\vec{p} = \overrightarrow{AB}$ and $\vec{q} = \overrightarrow{BC}$, express the following vectors in terms of $\vec{p}$ and $\vec{q}$: (a) $\overrightarrow{DE}$, (b) $\overrightarrow{AD}$, (c) $\overrightarrow{CD}$.

2. (a) An airplane flying at a speed of 700 km/h on a bearing of 110° is heading into a storm blowing from the SE with a wind speed of 150 km/h. What is the speed of the plane relative to the ground, and what is the bearing of the actual course of the plane?

   (b) In the same storm another plane is flying at 500 km/h relative to the air. In which direction does the pilot have to fly the plane in order to follow a course towards the West? How fast is this plane moving relative to the ground?

3. A traffic light (or 'robot') weighing 150 N is suspended above a road by two wires which make angles of 70° and 50° with the vertical. What are the forces on the anchors where the wire are attached? (Assume that the wires to have no weight, and use the principle that if an object does not change velocity, the resultant of all the forces on it is zero.)

4. A canoeist, who is able to paddle at a speed of 3 km/h, wants to cross a river that is 400 m wide and flows at a speed of 2 km/h.

   (a) If he paddles straight across, allowing himself to be carried downstream, at what speed does he move, and how far downstream will he reach the other side? How long will the crossing take?

   (b) If he needs to reach a landing site straight across from where he is setting off, in which direction does he have to head? How long will it take to reach the landing site?

   (c) If there is a waterfall 160 m downstream (and there is no place further upstream to put his canoe into the water,) what is the minimum speed at which he has to paddle to avoid being carried over the waterfall?
   (Hint: draw the vector representing the velocity of the water, and the direction of the resultant so that he reaches the other side just before the waterfall, and use geometrical considerations to decide on the direction of the shortest vector for his velocity.)

5. The vertices of the quadrilateral $ABCD$ have position vectors $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, and so on. Write down the position vectors of the midpoints M, N, P and Q of the four sides, AB, BC, CD and DA, in terms of the position vectors of the vertices. Hence show that $MNPQ$ is a parallelogram, whatever the shape of the original quadrilateral.

6. $ABCD$ is a parallelogram, in which $\vec{p} = \overrightarrow{AB}$ and $\vec{q} = \overrightarrow{AD}$. The point M is the midpoint of side AD and the point N lies on BC such that $BN = \frac{1}{3}BC$. If the line MN intersects the line AB in the point P, find the value of λ such that $AP = \lambda\, AB$.
   (Hint: express $\overrightarrow{MN}$ in terms of $\vec{p}$ and $\vec{q}$, and let μ be such that $MP = \mu\, MN$.)

Products of Vectors

7. If $OABC$ is a rectangle, express the vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$ along the two diagonals in terms of $\vec{a} = \overrightarrow{OA}$ and $\vec{c} = \overrightarrow{OC}$. Simplify the scalar product of the two vectors you found, and deduce that if the diagonals are perpendicular, the rectangle is a square.

4 Matrices and Transformations

Linear transformations

1. Write down the matrices of the following linear transformations of the plane:

   T_1: reflection in the line $y = -x$,

   T_2: clockwise quarter turn about the origin,

   T_3: enlargement by a factor 3,

   T_4: turn through $180°$ followed by reflection in the x-axis,

   T_5: reflection in the x-axis followed by turn through $180°$,

   and hence calculate the effect of each one on the vectors $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

2. Given the three matrices $A = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0.5 \\ 3 & 1 \end{pmatrix}$, and the two vectors $\vec{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$, calculate the six products of one of these matrices with one of these vectors.

3. (a) The two columns of a 2×2 -matrix M_T describe into which 2-dimensional vectors the linear transformation T transforms the 2 base-vectors $\vec{i}$ and $\vec{j}$. So what kind of linear transformations are represented by the following three matrices?

   $$P = \begin{pmatrix} -1 & 3 & 2 \\ 0 & 2 & 4 \\ 1 & -2 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} -1 & 3 & 6 \\ 1 & -2 & 2 \end{pmatrix} \quad R = \begin{pmatrix} -1 & 3 \\ 5 & 1 \\ 4 & -2 \end{pmatrix}.$$

   (b) Find all the possible products of one of these matrices with one of the following two vectors:

   $$\vec{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

4. (a) Find the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$.

   (b) Find the matrix $M_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the linear transformation T which transforms the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ into the vector $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ into $\begin{pmatrix} 8 \\ -7 \end{pmatrix}$.

5. What linear transformations are represented by the following matrices?

   $$M_1 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad M_4 = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad M_5 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

   (Hint: Investigate their effects on the vectors $\vec{i}$, $\vec{j}$ and $\vec{i} + \vec{j}$.)

6. The image of the point $P(x,y)$ under the linear transformation T is the point $P'(X,Y)$ such that $\begin{pmatrix} X \\ Y \end{pmatrix} = M_T \begin{pmatrix} x \\ y \end{pmatrix}$. (Note that linear transformations apply to vectors, so to find the image of a point we transform its position vector.)

(a) What is the Cartesian equation of the line ℓ on which the points P with coordinates $x = t, y = 3t - 2$ lie, for all $t \in \mathbb{R}$?

Find, in terms of t, the image P' of the point $P(t, 3t-2)$ under the linear transformation T with matrix $M_T = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$.

The line ℓ' on which the points P' lie, for all $t \in \mathbb{R}$, is called the image of the line ℓ. What is the Cartesian equation of ℓ'?

(b) Take any two points A and B on the line ℓ in part (a), and find their image points A' and B' under the same transformation T. What is the equation of the straight line through A' and B' ?

(c) Similarly, what is the image of the line $y = 2x$ under that transformation T ?

Combining linear transformations

7. Write down the matrices representing the following linear transformations:

S: rotation through $90°$,
T: reflection in the line $y = -x$, and
$T \circ S$: S followed by T.

Now calculate $M_S M_T$ and $M_T M_S$, and compare them with $M_{T \circ S}$.

8. Calculate all the possible products between the following matrices.

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 0 & 4 & -2 \end{pmatrix} \quad B = (\,2\,) \quad C = \begin{pmatrix} 1 & 0 & 4 & -2 \\ 3 & -1 & 1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad E = (\,-1 \quad 3\,)$$

An $m \times n$ -matrix has m rows and n columns. State the condition for it to be possible to multiply together an $m_1 \times n_1$ -matrix and an $m_2 \times n_2$ -matrix.

9. (a) Given the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, calculate A^2, A^3 and A^4. Write down a formula for A^n, for $n \in \mathbb{N}$, and prove it by induction.

(b) Repeat (a) for the matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

10. Find the most general form of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which commutes with the matrix $P = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, i.e. find relations between a, b, c, d such that $MP = PM$.

11. The matrix $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents a rotation about the origin through angle θ, and the matrix $S(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ a reflection in the line $y = \tan\theta \cdot x$.

(a) By considering linear transformations, explain why $R(\theta) \cdot R(\phi) = R(\theta + \phi)$. Deduce two of the compound angle formulae.

(b) Multiply $S(\theta) \cdot S(\phi)$ and simplify the product. What linear transformation does the product represent?

(c) Multiply $R(\theta) \cdot S(\phi)$ and simplify the product. Again, what linear transformation does the product matrix represent?

12. Let $M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Calculate M^3, and describe the linear transformation represented by M.

(b) Describe the linear transformation represented by N, and hence find N^4.

Areas and determinants

13. The image of the triangle with vertices $A(2,1)$, $B(5,1)$ and $C(2,3)$ under the linear transformation represented by the matrix $M = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$ is $A'B'C'$.

(a) Find the coordinates of A', B' and C', and draw the original triangle and its image in one diagram.

(b) Find the area of the original triangle, and the determinant of the matrix M.

(c) Hence determine the area of the triangle $A'B'C'$.

14. (a) If A is a 2×2 -matrix and λ a scalar, show that $\det(\lambda A) = \lambda^2 \det A$.
What do you think is the corresponding identity if A is a 3×3 -matrix?

(b) If A and B are two 2×2 -matrices, show that $\det(AB) = \det A \cdot \det B$.

15. (a) Write down the matrix M of the linear transformation which transforms the triangle with vertices $O(0,0)$, $I(1,0)$, $J(0,1)$ into the triangle with vertices $O(0,0)$, $P(a,c)$, $Q(b,d)$. Use the determinant of M to write down the area of triangle OPQ.

(b) Use the result of part (a) to find the area of the triangle $O(0,0), P(2,5), Q(-3,1)$.

(c) Similarly, what is the area of triangle $K(3,-4), L(5,1), M(0,-3)$? (It may help you to draw the triangles in (b) and (c) in one diagram.)

(d) Finally, what is the area of the triangle $X(2,-2), Y(6,3), Z(5,-8)$? What might the significance be of the sign?

16. Compare the values of the following four determinants. Can you generalise?

$$\begin{vmatrix} 3 & 2 & 0 \\ 1 & -1 & 4 \\ -2 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 3 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & -1 & 4 \end{vmatrix}, \quad \begin{vmatrix} 2 & 3 & 0 \\ -1 & 1 & 4 \\ 1 & -2 & 2 \end{vmatrix}, \quad \begin{vmatrix} 3 & 2 & 0 \\ 1 & -1 & 4 \\ -1 & 0 & 6 \end{vmatrix}.$$

17. (a) Calculate the following 'triple product' of three vectors: $\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}\right) \cdot \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

(b) Evaluate the determinant $\begin{vmatrix} 1 & 0 & 4 \\ -2 & 3 & 1 \\ 1 & 2 & -2 \end{vmatrix}$

Can you explain the relation between your results? What do the two numbers you obtained mean in geometrical terms?

Inverse linear transformations and matrices

18. (a) Write down the matrix M_1 representing a rotation through $45°$ followed by an enlargement by a factor $\sqrt{2}$, and the matrix M_2 representing a rotation through $-45°$ followed by an enlargement by a factor $\dfrac{1}{\sqrt{2}}$. Show that the product of the two matrices is the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Write down the matrix N_1 representing an x-stretch by a factor 2 followed by a reflection in the line $y = x$, and the matrix N_2 representing an x-stretch by a factor $\frac{1}{2}$ followed by a reflection in the line $y = x$. Is the product of N_1 and N_2 also equal to I?

19. Using the matrices defined in problem 11. above, show that

$R^{-1}(\theta) = R(-\theta)$ and $S^{-1}(\theta) = S(\theta)$.

Express these identites as statements about the linear transformations represented by the matrices.

20. $M_T = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ is the matrix of the linear transformation T. Find the inverse matrix M_T^{-1}, and use it to determine which vector $\vec{u}$ is transformed by T into the vector $\vec{u}' = \begin{pmatrix} 15 \\ 3 \end{pmatrix}$.

21. Given the matrices $A = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, find the following matrices:

$AB, \quad BA, \quad A^{-1}, \quad B^{-1}, \quad (AB)^{-1}, \quad (BA)^{-1}, \quad A^{-1}B^{-1}, \quad B^{-1}A^{-1}.$

Which of these are equal? Try to formulate a general rule, and to prove it.

22. If $P = \begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 6 \\ 3 & -5 \end{pmatrix}$, solve the following three equations for the matrix X:

(a) $3P + 2X = Q$, (b) $PX = Q$, (c) $XP = Q$.

23. (a) Consider the following simultaneous equations: $\begin{aligned} 4x + 2y &= 7 \\ \lambda x + 3y &= a \end{aligned}$

i. Find the inverse of the matrix $\begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$, and hence solve the equations for $\lambda = 5$ and $a = 6$.

ii. For which value of λ will the equations fail to have a unique solution? If λ has that value, for which value of a will the equations have infinitely many solutions?

(b) Consider the following simultaneous equations: $\begin{aligned} 3x + y - z &= 2 \\ x + 2y + z &= a \\ x + \lambda y - 3z &= -5 \end{aligned}$

i. Use your calculator to find the inverse of the matrix $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -2 & -3 \end{pmatrix}$, and hence solve the equations for $\lambda = -2$ and $a = 3$.

ii. For which value of λ will the equations fail to have a unique solution? If λ has that value, for which value of a will the equations have infinitely many solutions?

24. Show that if the matrix M_T is non-singular (i.e. $\det M_T \neq 0$,) then there is no non-zero vector $\vec{u}$ such that $M_T\,\vec{u} = \binom{0}{0}$.

Which of the following statements logically follows from what you have just proved?

(a) If the matrix M_T is singular, then there is a non-zero vector $\vec{u}$ such that $M_T\,\vec{u} = \binom{0}{0}$.

(b) If there is a non-zero vector $\vec{u}$ such that $M_T\,\vec{u} = \binom{0}{0}$, then the matrix M_T is singular.

(c) If there is no non-zero vector $\vec{u}$ such that $M_T\,\vec{u} = \binom{0}{0}$, then the matrix M_T is non-singular.

25. For which value of λ will the following simultaneous equations have a non-trivial solution, i.e. a solution other than $x = y = z = 0$? Find the general form of that solution.

$$\begin{aligned} \lambda x - y + 2z &= 0 \\ -2x + 7y - 8z &= 0 \\ x + 3y - 3z &= 0 \end{aligned}$$

Manipulating information

26. (a) A corporation makes two products. Of product A they make 1000 per week, and each requires 4 hours of labour and 1 unit of materials; of product B they make 2000 per week, and each requires 2 hours and 3 units. How many hours of labour and how many units of material are needed each week?

    (b) Suppose the corporation has the choice between producing in the US, where one hour of labour costs \$ 8 and one unit of material \$ 10, and producing in the UK, where they have to pay \$ 6 per hour and \$ 12 per unit. How much does each product cost to make in each country?

    (c) What then would their total costs be in the US, and what in the UK?

    In each part, arrange the information given in the form of tables, and try to formulate your calculations in terms of matrix operations.

27. The table below shows the results of an athletics match between Achisadel School and Premsipim College.

	1st place	2nd place	3rd place
Achisadel	7	5	4
Premsipim	5	7	8

Two different methods of scoring were proposed.

	Method 1:	Method 2:
1st place	3 points	4 points
2nd place	2 points	2 points
3rd place	1 point	1 point

By multiplying two matrices, find whether the result of the match depended on the method of scoring chosen.

28. The airlines of three countries, Alphania, Betaria and Gammaria, own VC 10s, Tridents and Fokker Friendships. The matrix P shows how many aircraft of each type each airline owns. The matrix S shows the number of 1st- and 2nd-class seats in the three types of planes.

$$\begin{array}{c} \\ A \\ B \\ G \end{array}\begin{array}{c} \begin{matrix} V & T & F \end{matrix} \\ \begin{pmatrix} 2 & 3 & 0 \\ 1 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix} \end{array} = P \qquad \begin{array}{c} \\ V \\ T \\ F \end{array}\begin{array}{c} \begin{matrix} 1st & 2nd \end{matrix} \\ \begin{pmatrix} 35 & 100 \\ 10 & 88 \\ 0 & 52 \end{pmatrix} \end{array} = S$$

Calculate the following three products, and determine what information each one gives us:

$$P\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad S\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad PS\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

29. Mr. Adomaku and Madam Berko both own transport running between Kumasi and Takoradi. Their ‘fleets’ consist of lorries which carry up to 25 passengers, minibuses which carry up to 16 passengers, and taxis which carry up to 5 passengers.

On a certain day, Mr. Adomaku has 3 lorries, 2 minibuses and 4 taxis at Takoradi. Madam Berko has 4 lorries and 5 minibuses. Write this information as a matrix.

By a suitable matrix multiplication, find the total number of passengers that each fleet can carry from Takoradi to Kumasi that day, assuming that each vehicle makes just one journey.

On the same day, Mr. Adomaku has 2 lorries, 3 minibuses and 1 taxi at Kumasi, and Madam Berko has 3 lorries, 2 minibuses and 2 taxis. Write this information as a matrix.

By a suitable matrix addition and multiplication, find the total number of passengers that each fleet can carry that day.

30. Cupronickel and nickel silver are both alloys of copper and nickel. Nickel silver also contains some zinc. The proportions of copper, nickel and zinc in these alloys are given in the following matrix.

$$\begin{array}{l} \\ \text{Cupronickel} \\ \text{Nickel silver} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Cu} & \text{Ni} & \text{Zn} \end{array} \\ \begin{pmatrix} 0.75 & 0.25 & 0.00 \\ 0.80 & 0.15 & 0.05 \end{pmatrix} \end{array}$$

The densities of the three metals are as follows: Co: 8.93 g/cm^3 , Ni: 8.90 g/cm^3 , Zn: 7.14 g/cm^3 . Calculate the densities of cupronickel and nickel silver.

31. Two people, Alan and Beth, had a meal together consisting of cereal, soup and pop tarts. The first table contains information about how much they ate of each kind of food (in 100 g,) and the second some nutritional information about those foods (per 100 g.) The matrices with the numbers from the tables as elements are called P and Q:

$$\begin{array}{l} \\ \text{Alan} \\ \text{Beth} \end{array} \begin{array}{c} \begin{array}{ccc} \text{cereal} & \text{soup} & \text{tart} \end{array} \\ \begin{pmatrix} 0.5 & 2 & 0.5 \\ 0.5 & 1 & 1 \end{pmatrix} \end{array} = P \quad \text{and} \quad \begin{array}{l} \\ \text{energy(kcal)} \\ \text{protein (g)} \\ \text{fat (g)} \end{array} \begin{array}{c} \begin{array}{ccc} \text{cereal} & \text{soup} & \text{tart} \end{array} \\ \begin{pmatrix} 370 & 50 & 400 \\ 12 & 3 & 4 \\ 6 & 3 & 12 \end{pmatrix} \end{array} = Q.$$

(a) Consider the four ‘products’ PQ, P^TQ, PQ^T and P^TQ^T, (where M^T denotes the transpose of the matrix M.) Which of these four is, or are, defined? Which of them is meaningful?

(b) Calculate the meaningful one of the four ‘products’ in (a). Explain what kind of information this product gives.

(c) The cost of the three kinds of foods (per 100 g) was 25 p for the cereal, 20 p for the soup, and 40 p for the pop tart. Use a matrix product to calculate the costs of Alan’s and Beth’s meals.

(Note that problems 27 - 30 are not original.)

5 Sets, Relations and Groups

Sets and set operations

1. Simplify the following (where $\mathcal{E}$ denotes the universal set):

   (a) $A \cup \emptyset$ (b) $A \cap \emptyset$ (c) $A \cup \mathcal{E}$ (d) $A \cap \mathcal{E}$

   (e) $A \cup A$ (f) $A \cap A$ (g) $A \cup (A \cap B)$ (h) $A \cap (A \cup B)$

   (i) $A \cup A'$ (j) $A \cap A'$

2. (a) Use Venn diagrams to show that union and intersection are associative, i.e. that

   (i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

   (– which means that such expressions can be written without brackets.)

   (b) Use Venn diagrams to show that intersection is distributive over union, and that union is distributive over intersection, i.e. that

   (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   If we identify union of sets with addition of numbers, and intersection with multiplication, which of these two holds?

   (c) Use Venn diagrams to verify de Morgan's laws:

   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

3. The identities of set theory apply, of course, also to sets of outcomes in probability theory.

   Suppose the probability that a student chosen at random does well in Maths is 0.52, that they do well in Physics 0.44, and that they do well in both 0.26. Find the probability that they do well in one or both of those of subjects, and hence the probability that a student will do well in neither of them.

   Use the multiplication law $p(A \wedge B) = p(A) \cdot p(B|A)$, to find the probabilites that someone who has done well in Maths will also do well in Physics, and that someone who has not done well in Maths will also not do well in Physics?

4. Simplify the following, either using Venn diagrams or the laws of set operations:

   (a) $A \cap (A' \cup B)$ (b) $A \cap B \cap (A' \cup B')$

   (c) $(A \cup B)' \cup B'$ (d) $(A \cup B) \cap (A \cup B')$

   (e) $(B \cup C) \cap (C \cup A) \cap (A \cup B)$ (f) $(A \cup B') \cap (B \cup C') \cap (C \cup A')$

   (g) $(A \cup B) \cap (A \cup B' \cup C)$ (h) $(A \cup B) \cap (A \cup C) \cap (B \cup C) \cap C'$

5. (a) In a group of 160 students, 58 study HL Maths and 83 are boys. If 52 girls don't do HL Maths, how many boys are doing HL Maths?

   (b) Of 29 days, there are 14 E-days, 7 O-days and 9 T-days. If there are 4 days that are E and T, and 2 days that are O but neither E nor T, how many 'other' days are there, i.e. days that are not E, O or T?

   If moreover there are 3 days that are T but neither E nor O, but no day that is E, O and T, how many days are there that are E only?

Relations, equivalence relations

6. If $P = \{a, b, c\}$, $Q = \{b, c, d\}$ and $R = \{i, j, k\}$, show that
(i) $(P \times R) \cup (Q \times R) = (P \cup Q) \times R$ (ii) $(P \times R) \setminus (Q \times R) = (P \setminus Q) \times R$.
(Note that $X \setminus Y$ is defined as $X \cap Y'$.)

7. (a) The following relations are subsets of the product set $\mathbb{R} \times \mathbb{R}$. Represent them as sets of points in the Cartesian plane.

$R_1 = \{(x, y) |\ xy = 0\}$
$R_2 = \{(x, y) |\ x^2 + y^2 = 9\}$
$R_3 = \{(x, y) |\ (x - 2)(y - 1) < 0\}$
$R_4 = \{(x, y) |\ x^2 + y^2 < 4\}'$
$R_5 = \{(x, y) |\ 2x + y = 4\} \cap \{(x, y) |\ -x + y = 1\}$
$R_6 = \{(x, y) |\ x + y \leq 4\} \cap \{(x, y) |\ x \geq 1\} \cap \{(x, y) |\ y \geq 2\}$
$R_7 = \{(x, y) |\ x^2 \leq 1\} \cup \{(x, y) |\ y^2 \leq 1\}$

(b) Consider the following subsets of the product set $\mathbb{R}^2 \times \mathbb{R}$. How could one represent these relations graphically?

$P = \{((x, y), z) |\ z \geq 2x - y + 3\}$
$S = \{((x, y), z) |\ x = \cos z, y = \sin z\}$

8. State clearly the three properties of an equivalence relation $\sim$ on a set M. (Hint: an equivalence relation is r..., s... and t... .)

For each of the following relations, check if it is an equivalence relation. If it is not, explain why it is not; if it is, try to describe the equivalence classes:

(a) H is the set of all people now in the world, and for any $a, b \in H$, $a \sim b$ if

i. a and b are blood relatives,
ii. a and b have the same mother,
iii. a lives within 10 km of b.

(b) $E_1 = \{(k, l) \in \mathbb{N} \times \mathbb{N} \mid k$ has a common factor with $l\}$
$E_2 = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m - n$ is an even integer$\}$
$E_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \geq y\}$
$E_4 = \{((p, q), (s, t)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid s - p = t - q\}$,
where $\mathbb{R}^2 = \{$points $P(x, y)$ in the plane$\}$
$E_5 = \{(w, z) \in \mathbb{C} \times \mathbb{C} \mid w\bar{z} \in \mathbb{R}\}$, where $\bar{z}$ denotes the complex conjugate of z.

9. Given the following equivalence classes, try to suggest a set and an equivalence relation which partitions that set: {Asare, Karelle, Ivan}, {Kristin}, {Robert, Igor, Tomas, Jonathan, Kazi, James}, ..., {Miho}.

Functions

10. If $X = Y = \{1, 2, 3, 4\}$, which of the following subsets of $X \times Y$ are functions? Explain why those that aren't aren't. Which of the functions are injections, which are surjections?

    $A = \{(1, 3), (2, 2), (1, 1), (2, 4)\}$ $\quad B = \{(1, 2), (2, 3), (3, 2), (4, 3)\}$

    $C = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$ $\quad D = \{(4, 1), (3, 2), (2, 3)\}$

    Why can there be no surjection here that is not also an injection?

11. For each of the following relations, decide whether it is a function, an injection ('one-to-one'), a surjection ('onto'), a bijection. (H is the set of all human beings.)

    $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \text{ is a prime factor of } n\}$

    $\{(m, n) \in \mathbb{N} \times \mathbb{N} \mid n \text{ is a prime factor of } m\}$

    $\{(a, b) \in H \times H \mid a \text{ is the father of } b\}$

    $\{(a, b) \in H \times H \mid a \text{ is the son of } b\}$

    $\{((m, n), q) \in \mathbb{N}^2 \times \mathbb{Q} \mid q = \dfrac{m}{n}\}$

12. Suggest suitable subsets $M, A \subset \mathbb{R}$ so that the following relation is a surjection. Why does it fail to be a bijection?

    $\{(z, (r, \theta)) \in \mathbb{C} \times (M \times A) \mid \text{if } z = x + iy, \text{then } r^2 = x^2 + y^2, \tan\theta = \dfrac{y}{x}\}$

13. For each of the following relations, decide whether it is a function, an injection, a surjection, a bijection.

    $R_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 - y = 2\}$

    $R_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y^2 = 1\}$

    $R_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 - y^2 = 1\}$

    $R_4 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 2x - y = 1\}$

    $R_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x\, y - 3 = 0\}$

    In each case, find the 'largest' domain $S \subset \mathbb{R}$ and range $T \subset \mathbb{R}$ for the relation to be (i) a function, (ii) an injection, (iii) a surjection.

14. The functions $f, g : \mathbb{Z}^2 \to \mathbb{Z}^2$ are defined such that

    (i) $f(x, y) = (y - 1, x - 1)$ $\quad$ (ii) $g(x, y) = (y - x, y + x)$.

    Check for each function if it is a bijection, and if it is find its inverse function. Also find the function $g \circ f$ (i.e. f followed by g.)

15. If $f : A \to B$ and $g : B \to C$ are both surjections, it can be shown that $g \circ f : A \to C$ is also a surjection. Find an example to show that the converse is not true, i.e. an example where $g \circ f$ is a surjection but not both f and g are.

    Similarly, if $f : A \to B$ and $g : B \to C$ are both injections, it can be shown that $g \circ f : A \to C$ is also a injection. Find another example to show that the converse is not true, i.e. an example where $g \circ f$ is a injection but not both f and g are.

16. Given the functions $f, g : I\!R \to I\!R$ such that $f(x) = 3 - x$ and $g(x) = 2x + 1$, find the following functions:
$f \circ g, \quad g \circ f, \quad f^{-1}, \quad g^{-1}, \quad (f \circ g)^{-1}, \quad (g \circ f)^{-1}, \quad f^{-1} \circ g^{-1}, \quad g^{-1} \circ f^{-1}.$
Which of these are equal? Try to formulate a general rule, and to prove it.

17. Given the functions $f_1(x) = x,\ f_2(x) = \dfrac{1}{x},\ f_3(x) = 1 - x,\ f_4(x) = \dfrac{x-1}{x},\ f_5(x) = \dfrac{1}{1-x},$

(a) find the functions $f_1 \circ f_3,\ f_5 \circ f_1$; what is the special role of f_1?

(b) find the functions $f_2 \circ f_3,\ f_3 \circ f_2,\ f_3^{-1}$.

(c) find the function $f_4 \circ f_5$ and deduce what the function f_4^{-1} is.

(d) find the additional function $f_6 = f_2 \circ f_4$ and find its inverse.

18. (a) Can we say anything about $f(0)$, if f is
(i) an even function, (ii) an odd function?

(b) What can be said about the constants a and b, if the linear function $f(x) = ax + b$ is
(i) an even function, (ii) an odd function?

Binary operations and groups

19. The 'multiplication table' for the binary operation $*$ defined on the set $\{e, a, b, c, d\}$ is given:

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	d	e	a	c
c	c	b	d	e	a
d	d	c	a	b	e

Using the above table, determine $(a * b) * c,\ a * (b * c),\ (b * c) * d,\ b * (c * d)$.
Ascertain which group axioms are satisfied by the given set under the given operation.
Find two subsets from the above set which form a group under the operation $*$.

20. Draw up the group table for the set $\{f_1, f_2, ..., f_6\}$ in problem 17. above, under the operation $\circ$ (composition of functions, which you may assume to be associative.)
Show that the set under that operation forms a group. Is the group a commutative group?
Find all the subgroups of the given group. What is the order of each of these subgroups, (i.e. what is the number of elements in the set)?

21. In the set G of symmetry transformations of the square $ABCD$, let R denote an anti-clockwise rotation of $\pi/2$ radians about the centre O of the square, I the identity transformation, L a reflection in the axis AC, and M a reflection in the axis BD.
Form the combination table for the elements I, R^2, L, M.
Deduce that these elements form a group of order 4; (again, associativity may be assumed.)

State another set of elements of G which form a group of order 4 and write down its group table.

22. Write down the group tables for the following groups of order 4, and compare them with the tables in the previous problem:

(a) $\{0, 1, 2, 3\}$ under addition modulo 4.

(b) $\{1, 2, 3, 4\}$ under multiplication modulo 5.

(c) $\{1, 3, 5, 7\}$ under multiplication modulo 8.

23. (a) Show that H, the set of all non-singular 2×2 real matrices, forms a group under matrix multiplication, (which may be assumed to be associative.)

(b) Let M be a given matrix, and let A be the set of all elements of H that commute with matrix M.

i. Show that for any M, A is a subgroup of H.

ii. Explain why the condition that $MX = XM$ for all $X \in A$ is not sufficient for the subgroup to be commutative.

iii. If $M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, find the most general form of an element of A, and hence show that A is a commutative subgroup of H.

24. The binary operation $\otimes$ is defined on the set $\mathbb{R}^2$ by $(p, q) \otimes (r, s) = (pr, q + ps)$. What is the identity for this operation, and what is the inverse of (x, y), when it exists?
What restriction(s) must be placed on the elements of $\mathbb{R}^2$ for the set with that operation to form a group?

25. (a) If $G = \{0, 1, 2, 3\}$, show that G under the binary operation 'addition modulo 4' does form a group, but that under the binary operation 'multiplication modulo 4' it does not.

(b) Examine whether or not the set G under the binary operation $\triangle$ forms a group, where $a \triangle b = |\, a - b \,|$.

(c) How many permutations are there on G ?
Show that the set of permutations on G which map 1 onto itself forms a group. Find an isomorphism for this group.
Explain why the set of permutations which map 1 onto 2 does not form a group.

26. Write down the group table of the cyclic group of order 6 with identity element e and a generating element a.
State the orders of all the elements of the group. Is there another generator?
List all the subgroups of the given group. What kinds of groups are they and of what order are they?
Find an isomorphism for this group, i.e. another group with essentially the same group table.