Problems for IB HL Mathematics and SL Mathematical Methods

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Contents

1 Sets, Relations, Functions

1. (a) In this problem, $x \in \{1, 2, 3, ..., 12\}$. Write down the following sets by listing their elements:

i. $E = \{x \,|\, x \text{ is even}\}$, $O = \{x \,|\, x \text{ is odd}\}$, $P = \{x \,|\, x \text{ is a prime number}\}$, $G = \{x \,|\, x \leq 4\}$.

ii. $E \cap P$, $E \cap O$, P', $E \cup G$, $G \cup P'$.

(b) Represent the following set in a Venn diagram: $A \cup (A' \cap B)$. What is a simpler way of writing the same set?

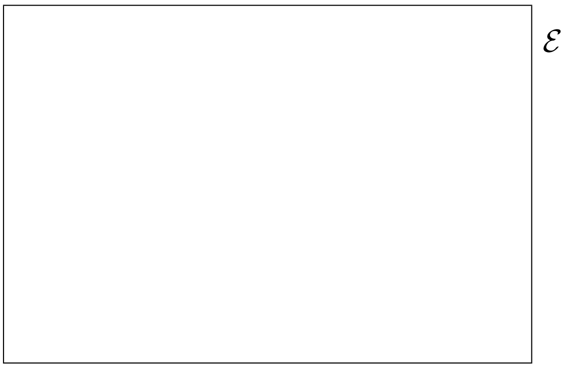
2. By drawing two clear Venn diagrams, show that $(A \cup B)' = A' \cap B'$.

3. By drawing two clear Venn diagrams, show that $(A' \cap B)' = A \cup B'$.

4. In the Venn diagrams provided, clearly mark the following sets; (note that A' denotes the complement $\complement A$ of the set A):

(a) $(A \cap B)' \cap C$,

(b) $(A' \cap C) \cup (B' \cap C)$.



5. On a typical cold evening at Atlantic College, with the heaters on in all buildings, 18 empty classrooms were checked in different buildings. The lights were on in 4 rooms and a window was open in 11 rooms (– and usually just above one of the radiators ...) If the lights were on *and* the windows open in 2 classrooms, in how many of the empty classrooms were the lights turned off and the windows closed?

6. Given that $A = B = \{1, 2, 3\}$, list the elements of the product set $A \times B$.
Also list the elements of the following three relations and represent them in three separate

lattice diagrams: $R_1 = \{(x,y) \in A \times B \,|\, y - x \geq 0\}$, $R_2 = \{(x,y) \in A \times B \,|\, y < x + 1\}$ and $R_3 = R_1 \cap R_2$.
Which, if any, of these relations are functions? Give reasons for your answer.
Try to describe the last of the relations in a simple way.

7. In this problem, $x, y \in \{-4, -3, -2, -1, 0, 1, 2\}$.

(a) List the ordered pairs in the relation $R_1 = \{(x,y) : \; x = |\,y|\}$.

(b) Write down the domain and the range of the relation R_1.

(c) List the ordered pairs in the relation $R_2 = \{(x,y) : \; y = x^2\}$.

(d) Write down the relations $R_3 = R_1 \cup R_2$ and $R_4 = R_1 \cap R_2$.

(e) Which of the four relations are functions, if any? For each one that you think is not a function, give your reason.

8. Given that $x, y \in I\!R$, represent the following relations in a coordinate system.

(a) $\{(x,y) : \; 2x + y = 3\}$

(b) $\{(x,y) : \; x - y \leq 1\}$

(c) $\{(x,y) : \; y = x^2 + 1\}$

9. State, with reasons, which of the following relations are not functions, and which, if any, are even or odd.

(a) $\{(x,y) \in I\!R \times I\!R : \; y = \dfrac{1}{x^2}\}$

(b) $\{(x,y) \in I\!R \times I\!R : \; |\,y| = \dfrac{1}{x}\}$

(c) $\{(x,y) \in I\!N \times I\!N : \; \dfrac{y}{x} \in I\!N \text{ and } x \neq 1 \text{ and } x \neq y\}$

10. Given functions f and g such that $f(x) = x - 4$ and $g(x) = 5x$, for all $x \in I\!R$, find the following fucntions:
$f \circ g, \quad g \circ f, \quad f^{-1}, \quad g^{-1}, \quad g^{-1} \circ f^{-1}, \quad (f \circ g)^{-1}, \quad (g \circ f)^{-1}$.
(Write these functions down in the same way as f and g are written above.)

What do you notice? (– in fact this result holds generally.)

11. Given the function $f : x \mapsto -2x + 3$, where $x \in I\!R$, write down $f(4)$ and $f \circ f(2)$, and solve the equation $f(x) = 7$.
Also find the general expressions for $f \circ f(x)$ and $f^{-1}(x)$.

12. Given the functions $f : x \mapsto 3x - 2$ and $g : x \mapsto x^2 - 4$, where $x \in I\!R$,

(a) write down the values of $f(1)$, $g(2)$, $f \circ g(3)$ and $g \circ g(4)$,

(b) determine the domains and the ranges of the two functions, and state whether either of them is even or odd, and

(c) find the following functions: $f \circ g$, $g \circ f$, f^{-1} and g^{-1} , simplifying your results where possible.

13. Given the functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 3x - 1$ and $g(x) = x^2 + 2$,

(a) find and simplify the functions $f \circ g$ and $g \circ f$,

(b) find the range of f and the range of g ,

(c) find the inverse function f^{-1} of f .

14. Given functions f and g such that $f(x) = 2x - 3$ and $g(x) = x^2$, for all $x \in \mathbb{R}$,

(a) find the inverse function f^{-1} of f,

(b) find the compound functions $f \circ g$ and $g \circ f$, and write down their values for $x = 5$,

(c) sketch the graphs of f, f^{-1} and g clearly in one coordinate system, and

(d) from your diagram estimate the coordinates of the point where the graphs of f and f^{-1} intersect each other.

15. Given the two functions $f(x) = \dfrac{1}{x-2}$ and $g(x) = x^2 + 1$, where $x \in \mathbb{R}$,

(a) state the greatest possible domains and ranges of f and of g,

(b) sketch the graph of g in a coordinate system,

(c) find the inverse functions f^{-1} and g^{-1} of f and g, and

(d) find the compound functions $f \circ g$ and $g \circ f$.

2 Algebra, Exponential and Logarithm Functions

1. Solve the equation $2^{3x} = 8\,(4^{x-\frac{1}{2}})$ for x.

2. Simplify the following expressions:

(a) $\dfrac{\sqrt{8} \times 4^{\frac{1}{3}}}{16^{\frac{1}{6}} \times 2}$ (Hint: convert into powers of 2.)

(b) $\dfrac{p^{\frac{5}{6}}\, p^{-\frac{1}{3}}}{\sqrt{p}}$

(c) $\dfrac{2+\sqrt{2}}{2-\sqrt{2}}$

3. (a) Simplify the following expressions:

i. $\log_8 32 + \log_8 4 - \log_8 2$,

ii. $\log_8 128 - 2\log_8 4$.

(b) Guess an integer approximation to the solution of the following equation, indicating your reasoning, and then use logarithms to find the solution (accurate to 3 significant figures, of course): $3^{x-1} = 28$.

4. (a) Evaluate i. $\log_2 32$ ii. $\log_2 \dfrac{1}{4}$ iii. $\ln e^3$.

(b) Evaluate $\log_9 15 + \log_9 6 - \log_9 10$.

(c) Express $\log \dfrac{a^2 b}{c^3}$ in terms of $\log a$, $\log b$ and $\log c$.

(d) Given that $3^x = 78$,

i. estimate the value of x correct to one s.f.

ii. find the value of x accurate to three s.f.

(e) Find x, given that $3^{2x-1} = 9$.

5. (a) Without using a calculator, simplify the following expression: $\log_2 \dfrac{\sqrt{3}}{2} - \log_4 12$.

(b) Guess an integer approximation to the solution of the following equation, indicating your reasoning, and then solve it: $5^{1-2x} = 120$.

(c) Solve the following equation for x: $\log_4 x + 8\log_x 4 = 6$.

6. (a) Simplify $2\log_5 10 - \log_5 2 - 2$, giving your answer precisely, as a single logarithm.

(b) Use logarithms to solve the equation $3^x\, 5^{x-1} = 8$.

7. (a) The line with equation $x + y = 1 + e$ cuts the graph of $y = e^x$ at P and the graph of $y = \ln x$ at Q. Sketch the line and the two curves in one diagram.
What is the distance between P and Q? (On the Maths Methods paper from which this comes students were not given the hint that a very simple guessing step is required.)

(b) Use logarithms to determine the set of integers n for which $(\frac{3}{4})^n \leq 0.1$.

3 Straight Lines, Linear Equations

1. Given the points $A(-2, 3)$ and $B(6, -1)$,

(a) determine the equation of the line through the two points,

(b) find the coordinates of the midpoint M of the line segment AB, and

(c) hence determine the equation of the perpendicular bisector, *i.e.* of the line through the midpoint M perpendicular to the line through A and B.

2. Given the points $A(-2, 1)$ and $B(4, 5)$, find the midpoint M and the perpendicular bisector of the line segment AB, and the distance between the two points.

3. A quadrilateral (i.e. a shape bounded by four straight lines) has vertices $A(-1, -3)$, $B(3, -1)$, $C(5, 3)$ and $D(-4, 3)$. Find the midpoints K, L, M and N of the sides AB, BC, CD and DA. Show that KL and NM have the same gradient and the same length. What kind of quadrilateral is $KLMN$ therefore?

4. Given the points $A(-2, 1)$ and $B(4, 3)$,

(a) calculate the distance between them,

(b) find the equation of their perpendicular bisector, *i.e.* of the straight line through the midpoint of the line segment AB perpendicular to the line AB.

5. (a) Given the two points $A(-2, 4)$ and $B(1, -2)$, find the distance AB.

(b) Find the equation of the line parallel to AB which passes through $P(2, 4)$.

(c) Also find the equation of the perpendicular bisector of the line segment AB.

6. Consider the triangle ABC with vertices $A(0, 3), B(4, 5), C(0, 13)$.

(a) What is the length of the side AB? (You do not need to use a calculator. Note that this length is the only number in the test which is not a 'nice' number.)

(b) Find the equation of the perpendicular bisector of the side AB, (*i.e.* of the line perpendicular to AB through its midpoint.)

(c) Find the equation of the median through A, (*i.e.* of the line through A and the midpoint of the opposite side, BC.)

(d) Find the point of intersection of the two lines you have found.

(e) Write down the equation of the line through A and C.

7. Solve the following sets of equations: (a) $\begin{aligned} 2x - 3y &= -8 \\ 3x + 2y &= 1 \end{aligned}$ (b) $\begin{aligned} 3x + 8y &= 4 \\ 2x - 3y &= 1 \end{aligned}$

8. Two straight lines have equations $2x - y = 4$ and $kx + 2y = -1$.

(a) If $k = 3$, find the coordinates of the point of intersection of the two lines.

(b) For which value of k do the lines have no point of intersection, i.e. for which value of k are they parallel?

(c) For which value of k are the two lines perpendicular?

9. (a) Given the points $A(-3,-2)$, $B(1,4)$ and $C(5,-4)$, find

i. the equation of the line AB, and the distance between A and B,

ii. the midpoint of the line segment BC, and hence the perpendicular bisector of that line segment.

(b) Solve the following simultaneous equations: 6x + 2y = 3
4x + 3y = 1

10. If two pads of paper and five pens cost £ 5.50, and three pads and 4 pens cost £ 6.50, what are the prices of one pad and of one pen?

11. A triangle has vertices $A(0,1)$, $B(8,-3)$ and $C(2,5)$.

(a) Find the gradients of the sides AB and AC. Comparing these gradients, what can you say about the directions of the sides? What kind of triangle is it therefore?

Find the lengths of the sides AB and AC, and use them to calculate the area of the triangle.

(b) What is the equation of the side BC, and at what point does it intersect the x-axis?

(c) Find the coordinates of the midpoint P of side AB and of the midpoint Q of side AC.

Hence find the equations of the medians through the vertices C and B, and the coordinates of the point G of intersection of the two medians.

Also find the averages of the coordinates of A, B and C. What do you observe?

12. The line ℓ_1 has equation $y = -2x + 1$ and P is the point with coordinates $(6,-1)$.

(a) Find the equation of the line ℓ_2 through P perpendicular to ℓ_1 .

(b) Find the coordinates of the point F of intersection of ℓ_1 and ℓ_2 .

(c) Hence determine the distance of the point P from the line ℓ_1 .

13. The line l has equation $y = \frac{1}{2}x - 2$ and P is the point with coordinates $(4,-5)$.

(a) Find the equation of the straight line perpendicular to l through point P, and the point of intersection of the two perpendicular lines. (Two lines are perpendicular, if the product of their gradients is -1.)

(b) Hence find the shortest distance of line l from point P, giving your answer precisely, as a root.

14. Given the points $A(2,-3), B(-2,5)$ and $C(5,6)$, find

(a) the equation of the line AB,

(b) the equation of the line through C which is perpendicular to AB,

(c) the coordinates of the point F of intersection of the two lines, and

(d) the distance between the points F and C.

(e) How far then is the point C from the line AB?

15. A triangle has vertices $A(-2,-2)$, $B(6,2)$, $C(3,3)$. (Note that all points you need in this problem have integer coordinates. Leave the distances you calculate as roots.)

(a) Find the equation of the line through A and B, and the distance between these two points.

(b) Find the equation of the median through C.

(c) Find the equation of the line ℓ through C which is perpendicular to AB.

(d) Find the point F of intersection of ℓ with the line AB, and the distance between C and F, *i.e.* the altitude or height of the triangle.

(e) Using some of your previous answers, calculate the area of the triangle.

16. (a) Solve the following equations simultaneously:

$$\begin{aligned} 4x - y &= 1 \\ 7x + 2y &= 4 \end{aligned}$$

(b) The graph of the function $y = ax^2 + bx + c$ passes through the three points $P(-1,6)$, $Q(0,3)$ and $R(1,4)$. By substituting the coordinates of these points into the given equation, obtain three simultaneous equations for a, b and c. Solve the equations to find a, b and c.

4 Quadratics, Polynomials, Rational Functions

1. Solve the following quadratic equations, giving all your answers precisely (i.e. not as an approximate decimal number.)

(a) $x^2 - 6x = 0$,

(b) $x^2 - 6x + 9 = 0$,

(c) $x^2 - 3x + 2 = 0$,

(d) $2x^2 - 3x - 2 = 0$,

(e) $3x^2 + 2x - 4 = 0$,

(f) $3x^2 + 2x + 4 = 0$.

2. Determine how many distinct real roots, or solutions, each of the following quadratic equations has, and solve the equation(s) that can be solved.

(a) $2x^2 - 3x - 2 = 0$.

(b) $-2x^2 - 3x - 2 = 0$.

(c) $x^2 - 6x + 9 = 0$.

3. For what values of k do the following quadratic equations have just one solution?

(a) $kx^2 - 12x + 9 = 0$

(b) $\dfrac{1}{x} = \dfrac{x}{kx - 9}$ (Hint: you have to rewrite the equation in the standard form first.)

4. Write down a quadratic equation with integer coefficients whose roots are $1/3$ and -2.

5. (a) Solve the quadratic equation $x^2 - 2x - 35 = 0$; (this can be done by factorisation.)

(b) Find the simplest quadratic equation with integer coefficients whose roots (solutions) are $\frac{2}{3}$ and 6.

(c) For which values of k does the equation $x^2 + 4x + k = 0$ have no (real) solutions.

6. (a) If the roots of the equation $2x^2 + 3x + 5 = 0$ are α and β, write down the sum and the products of the roots, and find the equation whose roots are $\dfrac{2}{\alpha}$ and $\dfrac{2}{\beta}$.

(b) For which value of p do the parabola $y = 2x^2+3x+5$ and the straight line $y = -x+p$ have just one common point, (i.e. the simultaneous equations have one solution)?

7. Sketch the graph of the parabola which crosses the $x-$axis at $(-1, 0)$ and $(3, 0)$, and the $y-$axis at $(0, \frac{3}{2})$, find its equation, and the coordinates of its vertex.

8. The curve with equation $y = ax^2 + bx + c$ passes through the origin $O(0, 0)$, and through the points $P(1, -1)$ and $Q(2, 2)$. By substituting the coordinates and solving simultaneous equations, or otherwise, find the values of a , b and c .

9. The graph of a quadratic function $y = ax^2 + bx + c$ passes through the three points $R(-2, 0)$, $S(4, 0)$ and $T(0, 4)$.

   (a) Find the values of a, b and c.

   (b) What are the coordinates of the vertex V of the parabola?

10. (a) If the straight line with equation $y = mx - 7$ passes through the point $A(3, 8)$, what is the value of m ?

    (b) Similarly, if the parabola with equation $y = ax^2 + bx - 2$ passes through the points $P(-1, 6)$ and $Q(3, 10)$, what are the values of a and b ?

11. When an object is thrown upwards at time $t = 0$ sec with velocity v_0 from an initial height s_0 above the ground, its height at time t is given by $s = -\frac{1}{2}gt^2 + v_0 t + s_0$; (use the approximation $g = 10$ m/sec^2.)
A diver jumps from a springboard 15 m above the surface of a pool with an upward velocity of 6.5 m/sec.

   (a) When does the diver fall past the springboard on the way down?

   (b) At what time does the diver reach the greatest height, and how high above the pool is that?

   (c) When will she hit the surface of the water?

12. A point P has coordinates $x = -t + 1$ and $y = 2t - 1$.

    (a) By eliminating t from the equations for the x- and the y-coordinates, find the equation of the straight line on which P will lie for all values of t.

    (b) Find a formula in terms of t for the distance d between $P(-t + 1, 2t - 1)$ and the point $C(5, 6)$. Let s be the square of that distance.

    (c) Write s as a simple quadratic function of t. For which value of t is s a minimum? For that value of t, what are the values of s and of d ?

13. A parabola has equation $y = x^2 - 4x + 1$.

    (a) Sketch the parabola, labelling the points of intersection with the axes and the vertex clearly.

    (b) Find the coordinates of the points of intersection of the parabola and the straight line with equation $y = 2x - 4$.

14. (a) Sketch the graph of the function $y = -x^2 + 4x - 3$ in a large diagram, labelling the points of intersection with the axes and the vertex clearly. (Note that most of the diagram will be below the x-axis, so leave enough space.)

    (b) In the same diagram sketch the graph of the function $y = 2x - 6$.

(c) Find the coordinates of the points of intersection between the two graphs and mark them in your diagram.

15. (a) In a clear diagram, sketch the parabola with equation $y = 2x^2 - 4x - 6$, marking all the significant points on the graph.

(b) In the same diagram, sketch the line with equation $y = -4x + 2$.

(c) Calculate the coordinates of the points of intersection between the parabola and the line, and mark them in your diagram.

(d) For each value of k, $y = -4x + k$ is the equation of some straight line. For which value of k does this line just touch the parabola, i.e. for which value of k do the parabola and the line have just one common point?
(Method: Eliminate y from the equations of the parabola and this line; there will still be a k in the resulting quadratic equation. If you write down the discriminant of that quadratic equation, it will also still have a k in it. You can then find k by using the condition that there should be only one solution.)

16. Sketch the parabola with equation $y = x^2 - 2x - 8$ in a large diagram, labelling the points of intersection with the axes and the vertex clearly.

For what value of k is the straight line with equation $y = 2x + k$ a tangent to the parabola, *i.e.* for what value of k do the parabola and the straight line have precisely one common point? Sketch the tangent in the same diagram as the parabola.

17. (a) By first finding the intersections with the axes and the coordinates of the vertex, sketch the graph of the function $y = x^2 + 2x - 8$.

(b) What are the coordinates of the points where the line with equation $y = x - 2$ cuts the parabola in part (a) ? Sketch the line in the same diagram, and mark the points of intersection.

(c) For what value of k is the line $y = -2x + k$ a tangent to the parabola in part (a) ? (Hint: solve the equations of the parabola and the line simultaneously, and use the condition for the resulting quadratic equation to have only one solution.) Sketch the tangent in the same diagram.

(d) Find the coordinates of the common point of the parabola in part (a) and the parabola with equation $y = x^2$.

18. Consider the parabolas with equations $y = x^2 + 3x + 2$ and $y = -x^2 - x + c$.

(a) If $c = 8$, sketch the graphs of the two functions in one large, clearly labelled diagram, and find the coordinates of the two points of intersection of the two parabolas.

(b) Find the value of c for which the two parabolas just touch, i.e. for which they have one common point. In the same diagram as before, sketch the graph of the second parabola for that value of c.

19. Two parabolas have equations $y = x^2 + 1$ and $y = -x^2 + 4x + p$.

(a) If $p = 17$, find the coordinates of the points of intersection of the two parabolas. Draw a very rough sketch of the parabolas: estimate the intersections with the x-axis from the calculator.

(b) For which value of p do the parabolas touch one another, i.e. for which value of p do they have one common point?

20. (a) Sketch the graph of the function $y = \frac{1}{2}x^2 + x - 4$, marking the intersections with the axes and the vertex clearly.

(b) From your graph determine the set of values of c for which the equation $\frac{1}{2}x^2 + x - 4 = c$ has two different solutions for x.

(c) Find the points of intersection of the parabola with the line $y = -x + 2$ and sketch that line in the same diagram, marking the points of intersection.

(d) Find the value of k so that the line $y = -x + k$ is a tangent of the parabola, *i.e.* so that the line has only one point in common with the curve.

21. (a) Sketch a graph of $y = -x^2 + 4x + 5$ in a large diagram, making sure you label all the significant points clearly.
For which value of $k \in \mathbb{R}$ does the straight line with equation $y = 2x + k$ have only one common point with the parabola? Sketch that line in the same diagram.

(b) By substituting $u = x^2$ and first solving the resulting quadratic equation for u, solve the following equation for x: $x^4 - 2x^2 - 8 = 0$.

22. (a) Sketch the graph of the function $y = x^2 - 6x + 5$, labelling all significant points in your diagram carefully.

(b) Find the coordinates of the points of intersection of the graph in part i. and the line with equation $y = -x + 11$.

(c) For what value of k is the line $y = -x + k$ a tangent to the curve, *i.e.* for what value of k do the curve and the line have precisely one common point? Draw this line into your diagram.

23. (a) What is the remainder when the polynomial $P(x) = x^3 - 7x + 6$ is divided by $(x + 2)$? (Note that the first term of that polynomial is not x^2 ...)

(b) Guess, or otherwise find, one solution of the equation $P(x) = 0$, and hence factorise the polynomial completely.

24. Let $P(x)$ be the polynomial $2x^3 - 5x^2 - 11x - 4$.

(a) Without using polynomial division, find the remainder when $P(x)$ is divided by $(2x - 1)$.

(b) The equation $P(x) = 0$ has one solution which is a small integer. Find that solution by trial and error. By dividing $P(x)$ by a suitable factor and equating the resulting quadratic to zero, obtain the other two solutions.

(c) Calculate the sum and the product of the three solutions you found in (b). How might they be related to the coefficients in the polynomial?

25. (a) What is the remainder when $x^4 + 3x^3 + 2x^2 + 4x + 5$ is divided by $x + 1$?

(b) Simplify the following expression: $\dfrac{(x+1)^{\frac{3}{2}} - (x+1)^{\frac{1}{2}}}{(x+1)^{\frac{1}{2}}}$.

26. (a) What is the remainder when $x^4 + 3x^3 + 2x^2 + 4x + 5$ is divided by $x + 1$?

(b) For what value of k is $(x - 2)$ a factor of the polynomial $2x^3 + 3x^2 - 8x + k$?
For that value of k, factorise the polynomial completely.

27. (a) The polynomial $P(x) = 2x^4 + 3x^3 + ax^2 + bx + 7$ has a remainder of 15 when divided by $(x - 1)$, and a remainder of -3 when divided by $(x + 2)$. What are the values of a and b ?

(b) Divide the polynomial $4x^3 + 8x^2 + x - 2$ by $2x^2 + x - 1$.

28. (a) Sketch the graph of $y = \dfrac{2x+3}{1-x}$, labelling the intersections with the axes and the asymptotes clearly.

(b) If the graph of $y = \dfrac{x+b}{x+d}$ has a vertical asymptote at $x = 2$ and crosses the x-axis at $x = -3$, what are the values of b and d ?

29. The vertical asymptote of the graph of the function $y = \dfrac{ax+2}{x+b}$ is $x = 2$, and it intersects the x-axis at $x = -3$.

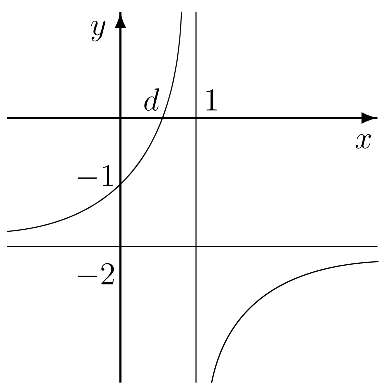
(a) What are the values of a and b ? (These values are not needed for part (b).)

(b) Write down

i. the equation of the vertical asymptote and
ii. the intersection with the x-axis

of the graph of the reciprocal function $y = \dfrac{x+b}{ax+2}$.

30. The diagram on the right shows the graph of a function $y = \dfrac{ax+b}{x+c}$. What are the values of a, b, c and d ?



31. Write down the equation of the horizontal asymptote of the graph of $y = \dfrac{x^2}{(x+1)(x-3)}$, and find the x-coordinate of the point where the graph intersects that asymptote.

Determine the set of y-values that the function cannot take. (Hint: rewrite as a quadratic equation for x and use the condition that there should be no real solution.)

5 Inequalities, Linear Programming

1. For what range or ranges of x-values do the following inequalities hold?

   (a) $4 - 3x > 10$

   (b) $2x^2 - 3x \leq 2$ (Hint: Write as a quadratic inequality and factorise.)

   (c) $\dfrac{x-2}{2x+5} > 0$

2. In one coordinate system, sketch the two straight lines with equations $y = -x + 4$ and $y = 2x - 5$.

   Mark the area in the coordinate system where $y \geq -x + 4$ and $y \leq 2x - 5$.

   Which point in the area you marked is closest to the origin? What is the distance of that point from the origin?

3. A manufacturer makes two varieties, V and W, of an article having parts that must be cut, assembled, and finished; the manufacturer knows that as many articles as are produced can be sold. An article of variety V takes 25 minutes to cut, 60 minutes to assemble, and 68 minutes to finish; it yields £30 profit. An article of variety W takes 75 minutes to cut, 60 minutes to assemble, and 34 minutes to finish, and yields a £40 profit. Not more than 450 minutes of cutting time, 480 minutes of assemb

   (a) One of the constraints on x and y is $25x + 75y \leq 450$. Write down the other constraints as inequalities, and represent the set of feasible solutions as a region in an xy-plane.

   (b) Write down an expression for the profit P in terms of x and y. How many articles of each variety should be manufactured each day to maximise profit?

   (c) When articles of the two varieties are manufactured so as to maximise profit, the time of one of the three departments – cutting, assembling or finishing – is not fully used. Determine from your graph which department it is, and calculate at what percentage of full capacity it is working.

6 Sequences, Series

1. (a) Rewrite in $\Sigma-$notation: $5+8+11+14+...+50$.

   (b) Evaluate separately the following expressions, showing clearly what terms you are adding, and compare the results.
   i. $\Sigma_{r=3}^{5}\,(r^3-2\,r^2)$ ii. $\Sigma_{r=3}^{5}\,r^3-2\,\Sigma_{r=3}^{5}\,r^2$

2. The three numbers $x+2,\ 3x-1,\ x^2+2$ are the first three terms of an A.P.. By using the definition of an A.P., or otherwise, find the possible values of x, and in each case write down the first four terms of the A.P.

3. (a) The numbers $x+1$ and $2x$ and x^2-5 are the first three terms of an A.P.. Find the two possible values of x, and for each of those values the first term and constant difference of the sequence.

   (b) An A.P. has first term 3 and constant difference -2. What is the sum of the first 20 terms of this series?

4. (a) -3 and 6 are the first two terms of an arithmetic progression. Write down the constant difference, and calculate the 23rd term, and the sum of the first 23 terms.

   (b) Which term a_n of this A.P. is equal to 276 ?

5. -3 and 6 are the first two terms of a geometric progression. Write down the common ratio, and the next two terms. Calculate the 10th term, and the sum of the first ten terms.

6. (a) The third term of a G.P. is 10 and the seventh term is 40. Find the two possible values for the common ratio, and the first term.

   (b) A G.P. has a common ratio of $\frac{2}{3}$ and a sum to infinity of 108. What is the first term, and what is the sum of the first 3 terms?

7. A geometric progression has first term 3 and its sum to infinity is 10. Find the common ratio r and the fourth term.

8. Write down the common ratio r and the sum of the first three terms of the following GPs:
   (a) $9-12+...$ (b) $9+6+...$
   Also for both of these series, write down the sum S_∞ to infinity, if it exists. (The sum to infinity of a GP is given by $S_\infty=\dfrac{a}{1-r}$. If it does not exist, state that it does not exist.)

9. (a) By using the definition of a G.P., or otherwise, find the values of $x\in I\!R$ for which $x-1$, $x+2$ and $3x$ form the first three terms of a G.P., and write down what the values of the first three terms are in each case.

(b) The sum of the first n terms of an A.P. is given by $S_n = 3n^2 + 2n$. Write down S_1 , S_2 and S_3 , and hence find the first three terms of the A.P., and its common difference.

(c) Use the formula, $S_n = \frac{n}{2}(2a + (n-1)\,d)$, for the sum of the first n terms of an arithmetic progression with first term a and common difference d, to find, in terms of n, (i) the sum N_n of the first n natural numbers (not including 0,) (ii) the sum E_n of the first n even natural numbers (not including 0,) and (iii) the sum O_n of the first n odd natural numbers.
Show that $O_n + E_n = N_{2n}$ and that $O_n + n = 2N_n$, and explain in simple terms why these relations should hold.

10. Calculate the sums of the following arithmetic progressions.

(a) $S_1 = 11 + 13 + 15 + 17 + ... + 99$,

(b) $S_2 = 15 + 21 + 27 + 33 + ... + 99$.

Hence find $S_3 = 11 + 13 + 17 + 19 + 23 + ... + 97$.

11. (a) The 2nd term of a geometric progression is 24 and its 6th term is 1.5 . Find the possible values of the common ratio and the first term, and of the sum to infinity.

(b) A piece of string of length 200 cm is to be cut into 5 pieces whose lengths form an arithmetic progression, with the longest piece being three times as long as the shortest. What should the lengths of the pieces be?

(c) By using the definition of an A.P., or otherwise, find the values of x so that the numbers x, 10 and x^2 are the 3rd, 4th and 5th term, respectively, of an arithmetic progression. Taking the positive value of x, what are the first term and the common difference of the series?

(d) Write down the 11th term of the sum: $\sum_{r=1}^{21}(\frac{3}{2} + \frac{r}{2})$, and evaluate the sum.

(You may find the following formula for the sum of the first n terms of an A.P. helpful: $S_n = \frac{n}{2}(2a + (n-1)d)$.)

12. A car cost \$ 12000 new. In the first year it loses 20% of its value, and in each year after that a further 10%. What is its value after 1, 2 and 3 years?
For $n \geq 2$, find a formula for its value after n years.

13. (a) An AP and a GP have the same first term 5 and the same second term, and the fourth term of the AP is the same as the third term of the GP. Set up two equations for the constant difference d of the AP and the common ratio r of the GP. What are the possible values of d and r ?

(b) Two people, Mr A and Miss G, plan to invest £2000 each in two different bank accounts.

i. If Mr A invests in a bank account that pays 6% simple interest per year, write down how much money there will be in the account after 1, 2 and 3 years. Find a general formula for the amount of money in the account after n years, and hence calculate how much money will be in the account after 10 years. When will the initial investment have doubled in value?

ii. If Miss G invests in an account that pays 4.5% compound interest per year, write down how much money there will be in the account after 1, 2 and 3 years. Find a general formula for the amount of money in the account after n years, and hence calculate how much money will be in the account at the end of the 10th year. At the end of which year will the initial investment have doubled in value?

iii. Another bank offers Mrs G an account, also paying compound interest, so that after 10 years she will have the same amount of money as Mr A does after 10 years. What interest rate is that bank paying?

14. According to legend, the inventor of chess was rewarded as follows: for the first square on the board, he was to receive 1 grain of rice, for the second square 2 grains, for the third square 4, for the fourth 8, and so on.
If the world's population is six billion (6×10^9) people, and a meal requires 200g of rice, and each gram of rice contains 20 grains, after how many squares would the total amount of rice have been enough for a rice meal for the whole population of the earth?

15. Find the constant term, *i.e.* the term without x in it, in the binomial expansion of $(2x - \frac{1}{x})^6$.

16. Find the constant term, *i.e.* the term without x, in the expansion of $(\sqrt{2}\, x - \frac{2}{x})^8$, simplifying your answer as much as possible.

17. Find the term with a^3 in the binomial expansion of $(2 + \frac{a}{2})^7$.

7 Trigonometry, Circles

Trigonometric Ratios, their Graphs

1. Evaluate the following trigonometric ratios, giving your answers accurately, *i.e.* without using a calculator:

   (a) $\cos 60°$,

   (b) $\sin 225°$,

   (c) $\tan \frac{5}{6}\pi$.

2. (a) On a sunny day, a vertical pole 8 m high casts a shadow of length 11 m on horizontal ground. What is the angle of the sun above the horizon?

(b) One angle in a right-angled triangle is 25°, and the opposite side has a length of 45 cm. How long are the other two sides?

3. A road goes straight up one side of a small hill and straight down the other side. The hill is 50 m high.

(a) On the way up, the road covers a horizontal distance of 600 m, as measured on a map. What is the angle that the road makes with the horizontal?

(b) On the way down, a car travelling along the road measures it to be 600 m long. What is the angle that the road makes with the horizontal on this side? Is it steeper or less steep than on the other side?
How long is the road down as measured on the map?

4. Find all the angles θ, where $0° \leq \theta \leq 360°$, such that

(a) $\sin\theta = -0.5$ (b) $\cos\theta = \dfrac{\sqrt{3}}{2}$ (c) $\tan\theta = 4$

5. By writing 15° as $45° - 30°$ and using the identity $\sin(A - B) = \sin A \cos B - \cos A \sin B$, find the precise value of $\sin 15°$ in the form $a\sqrt{6} + b\sqrt{2}$, where a and b are simple fractions.

   Use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to derive the precise value of $\cos^2 15°$, in the form $p + q\sqrt{3}$, where p and q are again simple fractions.

6. (a) Find the precise values, *i.e.* without using a calculator, of

   i. $\cos 135°$,

   ii. $\cot 210°$.

(b) Find the angles θ, where $0° \leq \theta \leq 360°$, such that $\tan\theta = -1$.

(c) Use your calculator to evaluate $2\sin 37° \cos 37°$, and then find the acute angle θ such that $\sin\theta = 2\sin 37° \cos 37°$. What general rule does this suggest?

7. (a) Without using a calculator, find the value of $2\sin 30^\circ \cos 30^\circ$ and the angle θ such that $\sin\theta$ equals that value. What general rule does this suggest?

(b) Using a calculator, find the value of $\cos 20^\circ \cos 50^\circ - \sin 20^\circ \sin 50^\circ$ and the angle θ such that $\cos\theta$ equals that value. What general rule does this suggest?

8. (a) If for some <u>acute</u> angle A, $\cos A = \frac{5}{13}$, find without using a calculator the values of $\sin A$ and $\tan A$.

(b) If for some <u>obtuse</u> angle B, $\sin B = \frac{1}{2}$, what is the value of $\cos B$?

(c) For what angles θ, where $0^\circ \leq \theta \leq 360^\circ$, is $\tan\theta = 10$?

9. Sketch the graphs of the following functions for $0^\circ \leq \theta \leq 360^\circ$ and state the lengths of their periods:

(a) $2\cos\theta - 2$, and

(b) $\sin 3\theta$.

10. Sketch the graph of $y = \cos(2x - \frac{\pi}{2}) + 1$ for $0 \leq x \leq 2\pi$.

11. For $0 \leq x \leq 2\pi$, sketch the graph of $y = 3\sin(\frac{x}{2} + \frac{\pi}{3}) + 2$, labelling all the intersections with the x- and y-axes accurate to 3 s.f..
What is the greatest y-value? For which value of x does the graph reach that highest point?

12. (a) i. Sketch the graph of $y = 2\sin(\frac{x}{2} - \frac{\pi}{6})$ for $0 \leq x \leq 2\pi$, marking the intersections with the axes clearly.

ii. Find the precise values of the solutions between 0 and 2π, inclusive, of the equation $2\sin(\frac{x}{2} - \frac{\pi}{6}) = 1$, and mark your solutions on the graph in (a).

(b) In one diagram, sketch the graphs of $y = 2\cos x$ and $y = x + 1$ for $-\pi \leq x \leq \pi$. Indicate on your graph how you could estimate the positive solution of the equation $2\cos x = x + 1$. Find the solution accurate to 3 s.f. using the calculator.

(c) Write down the greatest and least values of the function $y = 3\cos(2x - \frac{\pi}{4}) + \frac{3}{2}$.

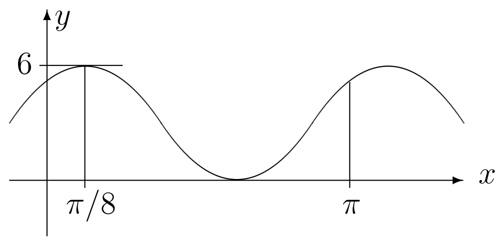
13. (a) Write down the period of the function $y = 2\sin\frac{1}{2}x - 1$. Sketch the graph of this function for $0 \leq x \leq 360^\circ$.

(b) i. In one diagram, sketch the graphs of the functions $y = \cos x$ and $y = \sin 2x$. Hence determine how many solutions the equation $\cos x = \sin 2x$ has in the range $0 \leq x \leq 2\pi$.

ii. Use the identity $\sin 2x = 2\sin x \cos x$ to find all the solutions of the equation $\cos x = \sin 2x$ in the range $0 \leq x \leq 2\pi$.

14. (a) In one diagram, sketch the graphs of $y = \cos x$ and $y = \sec x$, for $0 \leq x \leq 2\pi$, clearly labelling all intersections, asymptotes, and so on.

(b) The diagram on the right shows the graph of the function $y = a\sin(kx + \alpha) + c$.
Write down the period and the amplitude of the function. What are the values of a, k, α and c ?



Triangles, Circles

15. A triangle has sides of lengths 5 cm, 8 cm and 11 cm. Find the smallest angle in the triangle, and hence also the area of the triangle.

16. Three straight roads form a triangle between three cities: the distance from P-ville to Q-town is 7 km, from Q-town to R-burg 9 km, and from R-burg to P-ville 12 km.
Calculate the angle between the two roads from R-burg, to Q-town and to P-ville, and hence find the area enclosed by the three roads.

17. (a) The three sides of a triangle have lengths 10 cm, 15 cm and 20 cm. Find the value of the largest angle in the triangle, and hence the area of the triangle.

(b) From a boat out at sea, the angle of elevation (*i.e.* the angle between the horizontal and the line of sight) of the top of a cliff is 30°. When the boat has moved 60 m further away from the shore, the angle of elevation is 20°.
Draw a clear diagram of the situation, and find first the distance between the first position of the boat and the top of the cliff, and hence the height of the cliff.

18. In the triangle ABC the lengths of sides a and b are 17 cm and 20 cm, and the angle A is 40°. Find the possible values of the other two angles.

19. In the triangle ABC, angle $B = 30°$, side $b = 15$ cm and side $c = 20$ cm. Find the possible values of the other two angles.
Draw one sketch, clearly showing the two possible triangles.

20. A straight road 12 km long runs N from A-city to B-town; C-town lies 9 km to the NW of B-town, and there is another straight road between them.

How far is it from A-city to C-town as the crow flies, i.e. in a straight line?

Another straight road is proposed to run directly from A-city to C-town. Find the angle the new road would make with the old one from A-city to B-town.

The area enclosed by the three roads would become an industrial zone. If the value of land in the proposed industrial zone is £1.50 per m^2, what would the total value be of that area?

21. (a) If the distance between the moon and the earth is 382,100 km and the diameter of the earth is about 12,700 km, what angle approximately does the earth subtend at the moon? (For comparison, the moon apparently subtends an angle of 31' at the earth.)

(b) From a boat at a certain distance from the bottom of a cliff, the top of the cliff appears at an angle of elevation of 18°, (*i.e.* the line to the top of the cliff makes that angle with the horizontal.) When the boat has moved 30 m closer to the cliff, the angle of elevation is 24°. How high is the cliff?

(c) Calculate the greatest angle in the triangle whose sides have lengths 4 cm, 5 cm and 7 cm. What then is the area of the triangle?

22. A road runs along the straight line of a coast. From point A on the road, a buoy B out at sea is seen on a line which makes an angle of 40° with the road ahead. After driving 2 km along the road, to point C, the buoy is seen on a line which makes an angle of 25° with the road behind.

(a) Draw a clearly labelled diagram of road AC and the buoy B.

(b) How far from point A is the buoy?

(c) What then is the distance of the buoy from the coast?

23. The village Mura, the town Tokai and the capital Shuto are connected by three straight roads. The distance from Tokai to Shuto is 31 km, and from Tokai to Mura 29 km, and the roads to Mura and Tokai make an angle of 60° as they leave Shuto.

(a) Using the sine-rule, find the two possible angles which the roads to Shuto and Mura could make at Tokai, and hence the two distances which Mura could be from Shuto. What is the distance, accurate to 3 significant figures, between the two possible locations of Mura ?

(b) Using the cosine-rule, but not any of the results of (a), find the two distances which Mura could be from Shuto.
What is the precise distance between the two possible locations of Mura ?

(c) Given that no-one could walk in an hour from Shuto to Mura, what is the area enclosed by the three roads ?
If another straight road is to be built from Tokai, to meet the road between Mura and Shuto at right angles, how long will that new road be?

24. A-town, B-ville and C-burg are connected by straight roads. B-ville lies 8 km due East of A-town, and C-burg lies to the North-East of A-town, at a distance of 7 km.

(a) How far is C-burg from B-ville?

(b) On what compass bearing does one drive on the road from B-ville to C-burg?

(c) What is the area enclosed by the three roads between the three places?

(d) What is the shortest distance between C-burg and the road from A-town to B-ville?

25. A and B are two points on the circumference of a circle of radius $r = 12$ cm. If the distance AB is 10 cm, what angle does the chord AB subtend at the centre of the circle?

What then is the length of the minor arc AB?

26. (a) A and B are two points on the circumference of a circle of radius 10 cm. The line segment AB (called 'a chord') subtends an angle of 120° at the centre O of the circle, i.e. the angle $\widehat{AOB}$ is 120°. Find

    i. the length of the minor (= shorter) arc AB , and

    ii. the area of the smaller segment of the circle cut off by the chord AB .

(b) What angle (in radians) does a chord of length 24 cm subtend at the centre of a circle of radius 16 cm?
What is the area of the sector with that angle in that circle?

27. (a) A and B are points on the circumference of a circle of radius $r = 12$ cm and centre C, and the chord (line) AB subtends an angle of 75° at C. By finding the area of the sector and the area of the triangle ABC, find the area of the minor (smaller) segment cut off by AB.

(b) In another circle, of radius r, a chord AB subtends an angle of 30° at the centre C. Write down the formula, in terms of r, for the area of the minor segment cut off by AB.
If the area of that segment is 1.18 cm^2, find the radius r of the circle.

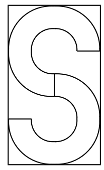
28. A and B are points on the circumference of the circle with centre O and radius 12 cm. If the length of the chord AB is $12\sqrt{3}$ cm, find the angle $\angle AOB$.
What then is the area of the smaller segment cut off by AB?

29. A and B are two points on the circumference of a circle of radius r and centre O. Find the angle $\angle AOB$, to the nearest degree, such that the perimeter of the segment cut off by the chord $\overline{AB}$ is equal to the diameter of the circle.
(The perimeter of a segment consists of two pieces, a chord $\overline{AB}$ and an arc $\overset{\frown}{AB}$.)

30. (a) One side of a triangle is of length 12 cm, and the angles at the vertices at the ends of that side are 45° and 60°. What is the area of the triangle?

(b) A circle has a radius of 20 cm, and a sector in that circle has an area of 500 cm^2. What is the length of the arc of the sector?

31. The logo of S-company is made up out of parts of circles, as shown in the diagram on the left. The radius of the inside circles is 5 cm and that of the outside circles 10 cm.
What is the area of the S-shape? (Hint: Draw a larger version of half of the S-shape, and calculate the difference between the areas of two sectors.)
What is the area of the enclosing rectangle, and what percentage of the whole area is inside the S-shape?



Trigonometric Identities and Equations

32. One of the four formulae below is valid for all angles A and B. By substituting values, or otherwise, determine which is the correct version. (There is no need to show all your working, as long as it is clear what you are doing.)

(a) $\sin(A+B) = \cos A \cos B + \sin A \sin B$

(b) $\sin(A+B) = \cos A \cos B - \sin A \sin B$

(c) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(d) $\sin(A+B) = \sin A \cos B - \cos A \sin B$

By writing $75° = 45° + 30°$ and then applying the correct version, calculate the precise value (in terms of roots) of $\sin 75°$.

33. Solve the following equations.

(a) $\cos^2 x = \frac{3}{4}, \quad 0 \le x \le \pi$.

(b) $\sin(2\theta - 40°) = -0.5, \quad 0° \le \theta \le 360°$.

(c) $\tan \frac{x+\pi}{2} = \sqrt{3}$, the general solution in radians.

34. Find the values of x, where $0 \le x \le 2\pi$, such that $2\sin^2 x + \cos x - 2 = 0$.

35. Find the general solution of the trigonometric equation: $2\sin^2 x + 11\cos x + 4 = 0$.

36. Using the identity $\tan^2\theta + 1 = \sec^2\theta$, find all the values of x, $0 \le x \le 2\pi$ such that $2\tan^2 2x + 3\sec 2x = 0$.

37. (a) Solve the equation $\cos\theta = -\frac{1}{2}$, where $0° \le \theta \le 360°$.

(b) Find the general solution of $\tan 2\theta = \sqrt{3}$.

(c) By using a suitable identity, rewrite the equation $2\cos^2 x + \sin x = 1$ as a quadratic for $\sin x$. Hence solve the equation, for $0 \le x \le 2\pi$, giving your answers in radians.

38. (a) Given that $\tan\theta = 2$ and that θ is a reflex angle, *i.e.* greater than 180° , find the precise values of the other trigonometric functions of θ .

(b) In one diagram, sketch the graphs of $y = \sin 2x$ and $y = \cos x$ for $0 \le x \le 2\pi$. Use your diagram to estimate the solutions in that range of the equation $\sin 2x - \cos x = 0$.

(c) Find the general solutions in degrees of $2\sin^2 x - 3\cos x = 0$.

39. Solve the following trigonometric equations. Give the solutions precisely, in the appropriate unit.

(a) $\sin 2x + 4\cos x = 0, \quad 0 \le x \le 2\pi$ (You may use $\sin 2x = 2\sin x \cos x$.)

(b) $2\cos(\theta - 20°) = -1, \quad 0° \le \theta \le 360°$

(c) $\cos^2\theta - \sin^2\theta = 0, \quad 0° \le \theta \le 360°$ (Hint: use that $\sin^2\theta + \cos^2\theta = 1$ for all θ.)

40. In one large, carefully labelled diagram sketch the graphs of the functions $y = \sin x + 1$ and $y = \cos 2x$, for $0 \le x \le 2\pi$.

From your graph determine the number of solutions of the equation $\sin x + 1 = \cos 2x$ in that range.

41. (a) In one diagram, sketch the graphs of the two functions $y = \sin 2x$ and $y = \cos x$ for $0 \le x \le 2\pi$. Hence determine the number of solutions in that range of the equation $\sin 2x - \cos x = 0$.

(b) Find the general solutions of the equation $\cos 2x - \sin x = 0$.

42. (a) Given that the angle A is obtuse, calculate the unknown side and angles of a triangle ABC in which $a = 5, b = 3$ and $B = 30°$. Also find the area of the triangle.

(b) It is given that $\cos\theta = \frac{17}{32}$. Using a double angle formula, or otherwise, but without using a calculator, find the possible values of $\cos\dfrac{\theta}{2}$.

43. Find the solutions of the following equations in the ranges specified.

(a) $\sin(3\theta - 30°) = -\dfrac{1}{2}$, where $0° \le \theta \le 180°$,

(b) $3\cos^2 x + \sin^2 x = 2$, where $0 \le x \le 2\pi$.

44. Rewrite the function $y = 2\cos x - \sin x$ in the form $R\cos(x + \alpha)$.
Hence, or otherwise, find the maximum and minimum values of the function, and the value(s) of x between 0 and 2π for which the function has a maximum.

45. By rewriting the function $f(x) = 4\sin x + 3\cos x$ in the form $R\sin(x + \alpha)$, or otherwise, determine (a) the greatest value of $f(x)$, and

(b) the smallest positive value of x for which $f(x)$ takes that greatest value.

46. (a) Rewrite $f(x) = \sin x + \cos x$ in the form $R\sin(x + \alpha)$ and hence solve the equation $f(x) = 1$, where $0° \le x \le 360°$.

(b) What are the greatest and the least values of $3\sin\theta - 2\cos\theta$?

47. Write down the greatest and the least values of the following functions:

(a) $y = 4\cos 2x - 3\sin 2x$;

(b) $y = (4\cos 2x - 3\sin 2x)^2 + 1$;

(c) $y = \dfrac{2}{4\cos 2x - 3\sin 2x + 7}$.

48. Find the solutions, between 0 and 2π, of the equation $\cos 2x + \cos x = 0$,

(a) approximately, by drawing two suitable graphs and finding their points of intersection;

(b) by using a suitable double angle formula; and

(c) by using the factor formula $\cos P + \cos Q = 2\cos\dfrac{P+Q}{2}\cos\dfrac{P-Q}{2}$.

49. (a) Find the general solution (in radians) of the equation $2\sin^2 x - \cos x - 1 = 0$.

(b) Use the identity $\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$ to find the precise values of θ such that $\sin 3\theta = \sin 2\theta$ and $0 \le \theta \le \pi$.

50. (a) Find the values of x, where $0 \le x \le 2\pi$, such that $\cos 2x = \sin x$.

(b) Rewrite $3\sin x + 4\cos x$ in the form $r\cos(x - \alpha)$.
For what values of k does the equation $3\sin x + 4\cos x = k$ have any solutions? Find the smallest positive solution if $k = 2.5$.

(c) Solve the equation $\cos 5\theta - \cos\theta = 3\sin\theta$, giving all solutions from 0° to 180°.
(The following identity may be useful: $\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$.)

51. Only one of the following two statements is a valid identity. Decide which one it is, and explain why you think the other one is not. Prove the one that is.
(a) $\cos(x+y)\ \cos(x-y) = \cos^2 x - \sin^2 y$, (b) $\cos(x+y)\ \cos(x-y) = \cos^2 x + \sin^2 y$.
(You may find the following two formulae useful: $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$.)

52. (a) By combining the compound angle formula $\cos(A+B) = \cos A\cos B - \sin A\sin B$ with the corresponding formula for $\cos(A-B)$, express $\sin A\sin B$ in terms of a sum or difference of trigonometric ratios.
Use a suitable substitution to derive a formula for $\cos P - \cos Q$, and use it to factorize $\cos 3x - \cos 7x$.

(b) Only one of the following four propositions is a valid trigonometric identity. Determine which one, carefully stating your reasoning.

i. $\cos 4\theta = 8\cos^4\theta + 8\cos^2\theta + 1$

ii. $\cos 4\theta = 8\cos^4\theta + 8\cos^2\theta - 1$

iii. $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

iv. $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta - 1$

53. Suppose that over the year, the time (on a 24-hour clock) of sunrise in a certain place is given by r and the time of sunset by s, where

$$r = 6 + 2\cos(\frac{\pi}{6}t - \frac{\pi}{6}) \quad \text{and} \quad s = 18 - 2\cos(\frac{\pi}{6}t + \frac{5\pi}{18}),$$

t being the time in months from the beginning of the year, (so that $t = 2$ on March 1, for instance, because 2 months have passed since the beginning of the year.)

(a) Sketch the function $r(t)$ for $0 \le t \le 15$. What is its period?

(b) Find the times of sunrise and of sunset on August 1, and hence the total time that the sun is up that day.

(c) Express, in terms of a single trig-function, the time between sunrise and sunset for any time t during the year. For what t is this function a minimum? What then is the date of the shortest day of the year?

You may find the following identity useful: $\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$
(– twice, in fact: for one problem it might be useful to substitute $Q = 0$ into this identity.)

Inverse Trigonometric Functions

54. (a) Find the greatest and the least values of the following functions:

i. $f(x) = 5\sin x - 12\cos x\,,$

ii. $g(x) = 5\sin x - 12\cos x + 7\,,$

iii. $h(x) = (5\sin x - 12\cos x + 7)^2\,.$

(b) Simplify the following expression, (*i.e.* write it without trig- or inverse trig-functions, as a simple polynomial in x): $\cos(2\arcsin x)\,.$

55. (a) Find the precise values of

i. $\cos(\arctan 2)$,

ii. $\arcsin(\cos\frac{\pi}{6})$. (Remember that $\cos\theta = \sin(90^\circ - \theta)$ for any θ.)

(b) Solve the following equation for x: $\arctan x - \arctan(x-1) = \arctan\frac{1}{7}$.

56. (a) Find the precise value of $\sec\arcsin\frac{1}{3}$.

(b) Find x such that $2\arctan 3 = \arcsin x$.

(c) Solve the equation $\arcsin x - \arccos x = \arcsin\frac{1}{2}$; to gain full marks, you must give the answer precisely.

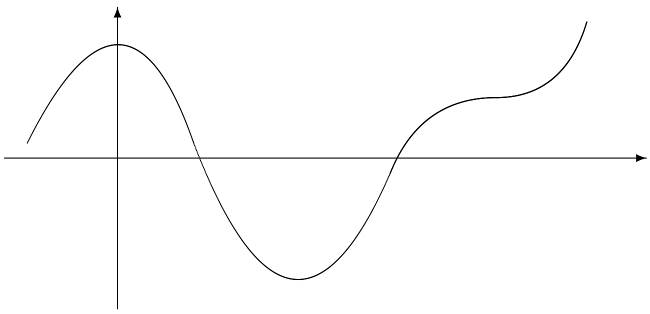
57. (a) Either using your calculator, or not, sketch the graph of $y = \cot x$, for $-\pi \leq x \leq \pi$. Hence sketch the graph of $y = \text{arccot } x$. What is the domain of this function? Suggest a suitable range.

(b) Find the value of x such that $\arctan 6 - \arctan x = \arctan 2$. (Only half the marks will be given if the answer is not the precise value.)

8 Differentiation

The Gradient Function, Methods of Differentiation

1. The top diagram on the right shows the graph of a function $y = f(x)$, with the same scale being used along the two axes.
In the bottom diagram, sketch the graph of the derivative or gradient function $y' = f'(x)$, using the same scale.
(Note that you have to hand in this sheet, with your name written on it.)



2. Differentiate the following functions w.r.t. (= with respect to) x.

(a) $y = (x+2)(x^2-3)$

(b) $y = \dfrac{x^4+5}{x^2}$

(c) $y = 6\sqrt{x}^3 + \dfrac{4x+2}{\sqrt{x}}$

(d) $V = k\,x^3$

3. Differentiate the following functions:

(a) $f(x) = (x+1)\sqrt{x}$.

(b) $f(x) = \dfrac{3-x}{x^2}$.

(c) $f(x) = 7(3x-2)^5$.

(d) $f(x) = 2\sin x - \cos(2x)$.

4. Differentiate the following functions with respect to the variable indicated and simplify:

(a) $y = \sqrt{x}(x + \dfrac{1}{x})$ (w.r.t. x,)

(b) $y = 3\sin x - 4\cos x + k^2$ (w.r.t. x,)

(c) $y = \arctan x + \dfrac{1}{x}$ (w.r.t. x,)

(d) $y = \ln e^{2x-3}$ (w.r.t. x,)

(e) $v = e^{u+2}$ (w.r.t. u,)

(f) $s = \frac{1}{2}gt^2 + v_0t + s_0$ (w.r.t. time t.)

5. Differentiate the following functions with respect to x and simplify your results.

(a) $y = 4\sqrt{x}$

(b) $y = \sin^2 x - \cos^2 x$

(c) $y = x^5 \ln x$

6. 'Eleven derivatives in fifteen minutes':

(a) $y = 7e^{2x-1}$

(b) $y = (x^2 - 3)^2$

(c) $y = \ln e^{3x}$

(d) $y = \dfrac{x^2 - 3}{x^3}$

(e) $y = x^3\sqrt{2 - x}$

(f) $y = \sin 2x \cos 3x$

(g) $y = 2\tan(x - \dfrac{\pi}{3})$

(h) $y = \dfrac{2x - 1}{x + 3}$

(i) $y = \ln(3x - 1)^2$

(j) $y = x^2\ e^{2x}\ \cos x$

(k) $s(t) = -\frac{1}{2}gt^2 + 10t + 3$
(Note that derivatives w.r.t. time t are denoted by a dot: $\dot{s}(t)$.)

7. Differentiate the following functions w.r.t. x and simplify your results.

(a) $y = -\ln \cos x$,

(b) $y = x^3 \sin 3x$,

(c) $y = \ln e^x$,

(d) $y = \dfrac{(x - 2)^3}{x^7}$.

8. Differentiate the following functions w.r.t. x and simplify your results.

(a) $y = \ln \tan x$,

(b) $y = x\ e^{(x^2)}$,

(c) $y = \dfrac{2x - 4}{\sqrt{x}}$,

(d) $y = \dfrac{(x + 3)^7}{x^2}$.

Tangents, Normals, Graphs

9. Find the equations of the tangent and of the normal to the graph of $y = 5 + 2x - x^2$ at the point where $x = 2$. Sketch the parabola and the two lines. (You are not required to find the intersections with the x-axis.)

10. (a) Differentiate the following functions with respect to x and simplify your results:

i. $y = e^{3-2x}$,

ii. $y = e^x \sin 2x$,

iii. $y = \dfrac{\ln x}{x}$,

iv. $y = \ln(\cos x)$.

(b) Find the equation of the tangent to the graph of $y = x\sqrt{x-1}$ at the point where $x = 2$.

11. Find the equation of the normal to the graph of $y = \frac{3}{2}x^2 - \frac{5}{2}x + 3$ at the point where $x = 1$.
Where does that normal intersect the two axes, and what is the area of the triangle bounded by that normal and the axes?

12. Find the equation of the normal to the graph of $y = \frac{1}{2}x^2 - x - 4$ at the point P where it cuts the x-axis and $x \geq 0$.

Then find precisely, or accurately to 3 s.f., the x-coordinate of the point Q where that normal cuts the curve again.

13. Consider the function $y = x^3 - 3x + 2$ and its graph.

(a) Find the coordinates of the point where the graph crosses the y-axis and determine, by trial and error, the coordinates of the two points which it has in common with the x-axis. (All the coordinates are simple integers.)

(b) Find the coordinates of the two turning points of the graph and determine their nature.

(c) Using the results of parts (a) and (b), sketch the graph of the function.

(d) Find the equation of <u>the normal</u> to the graph at the point where $x = 2$.

14. Use differentiation to find the stationary point on the graph of $y = x^2 + \dfrac{16}{x}$ and determine its type. Also find the intersection of the graph with the x-axis.
Use the calculator's graphing function to draw a sketch of the graph, marking the points you have found. (You should use the results that you calculated to set a window-size so you can see the important features of the graph.)

15. Use the product rule to differentiate the function $f(x) = (x-1)^3\,(x+3)$ and simplify your result. Hence find all the stationary points of the graph of $y = f(x)$ and determine

their nature. Sketch the graph of the function, marking the intersections with the axes clearly. (Hint: all the answers in this problem are integers.)

16. (a) Find the stationary points of the graph of the function $y = x^2 e^x$ and determine their nature.

(b) Consider the behaviour of the graph for large positive and negative values of x; (you may want to use your calculator for this.) Also find any intersections with the axes.

(c) Use the information you have to sketch the graph of the function.

17. Given the function $y = x^2\, e^{2x}$,

(a) find the point(s) of intersection of its graph with the axes,

(b) find the coordinates of the stationary points and determine their type, and

(c) find the x-coordinates of the points of inflection.

Hence sketch the graph of the function in a large diagram, clearly marking all the points you found above.

18. Determine the number n of stationary points (i.e. how many such points there are,) and the number m of points of inflection of the graph of $y = 3x^4 + 8x^3 - 5$.
If the function is defined only for the domain $\{x \in \mathbb{R} : -3 \le x \le 1\}$, what are the greatest and the least values, (i.e. the highest and the lowest values)?

19. Consider the function $f(x) = x \ln x$, for $x \ge 0$, and its graph.

(a) Find the stationary point of the graph, and determine what type it is.

(b) What happens to the gradient function $f'(x)$ near $x = 0$?

(c) Given that $f(x) \to 0$ as $x \to 0$, sketch the graph of $y = f(x)$ for $0 \le x \le 3$, showing clearly the behaviour near $x = 0$.

Applications: Maxima and Minima, Iterative Methods

20. (a) This part requires no differentiation.
Sketch the graph of $y = (x-1)(x-3)$, and hence find the greatest value and the least value it takes in the range $0 \le x \le 5$.

(b) When an apple is thrown vertically upwards at time $t = 0$ sec with initial velocity $v = 12$ m/sec, its height above the ground at time t is given by $s = -\frac{1}{2}gt^2 + 12t$, where g is a constant for which you should use the approximate value 10 m/sec^2 .
What are the height of the apple and its velocity 0.5 sec after it was thrown?
After how much time is the velocity of the apple 0 m/sec, and what is its height at that time?

21. The product of two positive numbers x and y equals 9. Write down an expression for $z = 4x + y$ in terms of x, and hence find the least (smallest) possible value of z.

22. x and y are two positive numbers whose sum is 24. What is the greatest value that the product z of one of these numbers with the square of the other one can be? (Hint: you must first obtain a relation between z and just one of the other variables.)

23. The graph of $y = x^3 - x + 4$ has two stationary points, a local maximum and a local minimum. Find (a) the precise y-value at the local maximum, and

(b) the greatest value of the function in the range $-2 \le x \le 2$.

24. A rectangular box with a square base of side x cm and height h cm is to be open at the top and have volume of 500 cm^3 .

(a) Write down an equation connecting x and h, and a formula for the surface area A cm^2 in terms of x and h .

(b) Deduce that $A = x^2 + \dfrac{2000}{x}$, and hence find the dimensions of the box which make the surface area a minimum.

25. A long straight road runs along a field, and the farm of Old MacDonald lies by the road. By walking along the road for 1 km, to a point A, and then at right angles to the road across the field for 1 km, Old MacDonald can reach one of his barns.

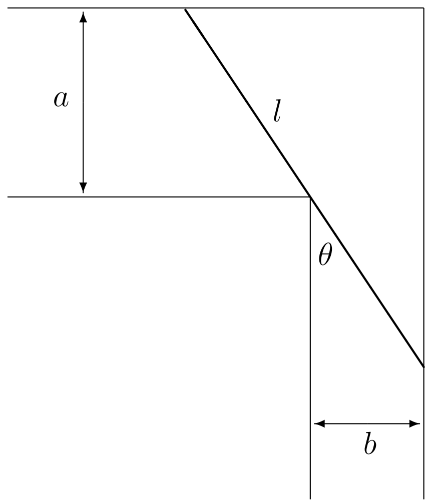
(a) If he walks at a speed of 5 km/h along the road and 3 km/h across the field, how long does it take him to walk from his farm to A along the road and then across the field to the barn? (You are reminded that velocity is distance over time.)

(b) How long would it take him to walk in a straight line across the field from his farm to the barn?

(c) One day, Old MacDonald decides to take a shortcut: he walks from his farm along the road to a point P, x km from A, and from there in a straight line across the field to the barn. Find an expression for the time that this walk takes him.

(d) Find the value of x that makes this time a minimum. What then is that shortest time?

26. A canal has at one point a right angled corner, its width being $a = 43.2\,m$ on one side of the corner and $b = 25\,m$ on the other. A pole of negligible thickness floating in the canal is to be manoevred around the corner in horizontal position.
By expressing the length l of the pole in terms of θ, a and b, as in the diagram, or otherwise, find the greatest possible length of the pole to 3 s.f.



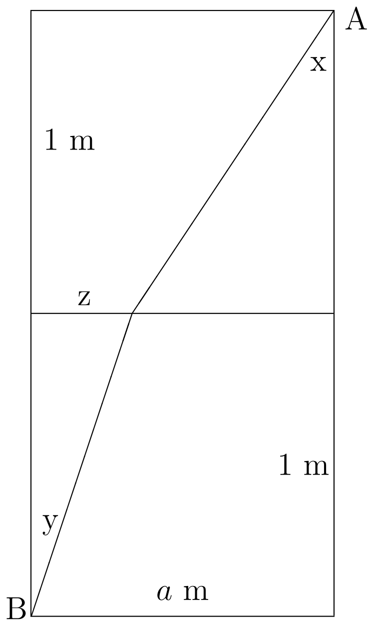
27. A mural (wall painting) 4 m high is painted on a vertical wall such that its bottom edge is 1 m above the eye level of a typical viewer.

(a) If a typical viewer stands at x m distance from the bottom of the wall looking up at the mural, the angle at her eye between the top and the bottom edge is θ. Using the formula $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$, show that $\tan\theta = \dfrac{4x}{x^2+5}$.

(b) Differentiate the function $t = \dfrac{4x}{x^2+5}$, and hence find the stationary point and determine its nature.
At what distance x does the typical viewer therefore have to stand from the wall to see the mural at its largest? What is that largest angle at which she can view the mural?

(c) Briefly explain why the value of x you found by differentiating t does give the greatest value of θ.

28. The diagram shows a cross section through a rectangular tank 2m high and a m wide; the top half is filled with a medium in which light travels at speed u, and the bottom half with a medium in which the speed of light is v. Light shines from the top right edge A of the tank to the bottom left B, crossing the interface a distance z from the side of the tank, and making angles x and y with the vertical sides at A and B. Note that y is a function of x.



(a) Express both $\tan x$ and $\tan y$ in terms of z, and hence find the simple equation relating $\tan x$ and $\tan y$.

(b) Differentiate the equation obtained in part (a) implicitly, to show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{\cos^2 y}{\cos^2 x}$.

(c) Write down the distance travelled in each medium in terms of x and y. Hence express the total time t that the light needs from A to B, in terms of x and y (and the velocities u and v of course).

(d) Find $\mathrm{d}t/\mathrm{d}x$, (remembering that y is a function of x,) and deduce a condition for the total time t to be a minimum.

(e) By substituting the result of part (b) into the condition in part (d) and simplifying, show that $\dfrac{\sin x}{\sin y} = \dfrac{u}{v}$ (Snell's Law.)

29. (a) Calculate the values of the function $f(x) = 2x^2 + x - 25$ for $x = 0, 1, 2, 3, 4$, and use these values to estimate the positive solution of $f(x) = 0$, accurate to one decimal place.

(b) Use the Newton-Raphson method once to improve your first estimate.

(c) Use the formula for the roots of a quadratic equation to determine the solution. What is the percentage error of the estimate you obtained in part (b) ?

30. Given that $1° = \frac{\pi}{180} = 0.01745$ (radians), use the method of small changes to find an approximation to the value of $\tan 43°$. What is the percentage error of the approximation? (You may, or may not, find the following formula helpful: $f(a + h) = f(a) + f'(a)h$.)

31. (a) Without using a calculator, find an approximation to the cube root of 7.76. Using the accurate value given by the calculator, what is the percentage error of the approximation?

(b) If the value of x is changed by $p\,\%$, find the percentage change of x^3 .
Hence, or otherwise, find the percentage change of $\sqrt[3]{x}$ if x is changed by $p\,\%$.

Reminder:
the formula for small changes is $\Delta y \approx \dfrac{\mathrm{d}y}{\mathrm{d}x}\Delta x$, and the relative change in x is $\dfrac{\Delta x}{x}$.

Implicit and Parametric Differentiation, Rates of Change

32. Given that $5x^2 - xy + y^2 = 7$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. Hence find the equation of the normal to the curve with that equation at the point where $x = 1$, $y = 2$.

33. Differentiate $4x(y+1) - \ln x + 2e^y = 7$ w.r.t. x. Hence find the normal to the graph given by that implicit function at the point $(1, 0)$.

34. Differentiate the equation $4y^2 - x^2y^2 - 9x^2 = 0$ implicitly with respect to x.
Hence find the equation of the tangent to the curve at the point $(1, \sqrt{3})$.

35. (a) Find the value of k such that the graph of the relation $(x^2+2)y+\sin y-4e^x+k=0$ passes through the point $P(0,\ \frac{\pi}{2})$.

For that value of k differentiate the relation implicitly w.r.t. x . Find the gradient of its graph at the point P and write down the equation of the tangent at that point.

(b) Differentiate the relation $y^4+y=x^6+x^4$ implicitly w.r.t. x and find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1,\ 1)$.

Differentiate the relation a second time implicitly w.r.t. x and find the value of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the same point.

36. Find the value of $k\in I\!R$ such that the graph of the relation $y^2\sin x+e^{2y}-x^2+k=0$ passes through the point $(\pi,1)$, and find the gradient of the graph at the given point.

37. A curve is given parametrically by $x=\cos t$, $y=\sin^2 t$. Find the gradient function $\mathrm{d}y/\mathrm{d}x$, in terms of the parameter t, and simplify it. Hence determine the x- and y-coordinates of the point where the gradient of the graph is 1.

38. A curve has parametric equations given by $\quad x=\dfrac{1+t}{1-2t},\quad y=\dfrac{1+2t}{1-t}$.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t , simplifying your answer as much as possible.

(b) Find the equation of the tangent to the curve at the point where $t=2$.

39. A curve is given in parametric form by $x=1+3\sin t$, $y=2-2\cos t$.

(a) Write down the greatest and least values of x and y .

(b) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t, and determine for which values in the range $0\le t\le 2\pi$ it is zero. Hence find the coordinates of the points where the curve has a horizontal tangent.

(c) Also find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t, and use it to determine the nature of the stationary points you found in b).

40. Find the gradient function $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, in terms of the parameter t, of the curve given by $x=2\sin t, y=3\tan t$.
Hence find the equation of the tangent to the curve at the point where $t=\dfrac{\pi}{6}$.

41. A curve is given parametrically by $x=\sin 2t,\ y=2\sin t$, where $0\le t\le 2\pi$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.
Find the x- and y-coordinates and the gradient of the curve for $t=0,\ \dfrac{\pi}{2},\ \pi,\ \dfrac{3\pi}{2},\ 2\pi$, and mark the five points and the direction of the curve at these points in a diagram. Hence sketch the graph of the curve.

42. Draw a large sketch of the curve with parametric representation $x = 3\sin\theta, y = 2\cos 3\theta$, for $0 \le \theta \le 2\pi$, marking clearly the minimum and maximum x- and y-values. In your diagram, mark the point A at which $\theta = \pi/4$.

Find the gradient function $\frac{\mathrm{d}y}{\mathrm{d}x}$, also in terms of θ, and hence calculate the gradient of the curve at A. Without finding its equation, draw the tangent at A.

Briefly explain

(a) why the graph of a function can never cross itself, and

(b) how it can happen that in the graph you have drawn, although x and y are both functions, the graph can cross itself.

43. Sketch the curve given parametrically by $x = 2\sin t, y = 3\sin 2t$. Write down the two values of the parameter t , $0 \le t < 2\pi$, for which (x, y) is the origin.
Find the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of the parameter t, and write down the gradients of the curve at the point where it crosses itself.

44. Air is being pumped into a spherical baloon at a rate such that the radius increases at a constant rate of 0.5 cm/min. If the baloon is empty to begin with, at what rate is the volume of the baloon increasing when the radius is 10 cm? At what rate does the air have to be pumped in when the volume has reached 4000 cm^3 ?
(The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.)

45. The two equal sides of an isosceles triangle have a length of 4 m and an angle θ between them. Write down an expression for the area of the triangle.
If the angle changes at a constant rate of 0.035 (which is about 2°) per day, what is the rate of change of the area of the triangle, in m^2/day, when $\theta = \frac{\pi}{3}$ (which is 60°)?
What then is the rate of change of the area in cm^2/hour?

46. A rocket R is rising straight up into the air, from a point P which is 3000 m away from an observer O in the same horizontal plane. The angle of elevation of the rocket is the angle $\angle ROP$ above the horizontal at which the rocket is seen by the observer.

When the angle of elevation is 30°, the rate of change of the angle of elevation is 0.1°/sec. What then is the speed of the rocket at that moment? (Remember to change the angles into the right unit!)

9 Integration

Methods of Integration

1. Do the following indefinite integrals:

(a) $\int (2x+3)^5 \, \mathrm{d}x$

(b) $\int \frac{5\sqrt{x}-2}{\sqrt{x}} \, \mathrm{d}x$ (Hint: Rewrite the function as a sum before integrating.)

(c) $\int (4\cos 2x + 3\sin \frac{x}{2}) \, \mathrm{d}x$

2. Find the following indefinite integrals and simplify your answers:

(a) $\int (e^{1-x}+1) \, \mathrm{d}x$,

(b) $\int x^2(7\sqrt{x}+6) \, \mathrm{d}x$.

3. (a) Do the following indefinite integrals:

i. $\int (2e^x + \frac{3}{x}) \, \mathrm{d}x$

ii. $\int \frac{5\sqrt{x}-2}{\sqrt{x}} \, \mathrm{d}x$ (Hint: Rewrite the function as a sum before integrating.)

(b) Evaluate the following definite integrals:

i. $\int_0^\pi (4\cos 2x + \sin \frac{x}{2}) \, \mathrm{d}x$ (You may be able to find the answer without integrating.)

ii. $\int_1^2 (2x-3)^4 \, \mathrm{d}x$

4. (a) Integrate the following two functions:

i. $f(x) = 6x^2 - \sqrt{x}$

ii. $f(x) = 2\cos \pi x + 4e^{2x}$

(b) Evaluate the following definite integrals:

i. $\int_{-2}^2 x(x+3) \, \mathrm{dx}$

ii. $\int_1^3 e^{1-x} \, \mathrm{dx}$

5. (a) Differentiate i. $y = \ln x^3$ ii. $y = 5\ \ln 2x$.

(b) Evaluate $\int_2^3 \frac{4}{2x-1} \, \mathrm{d}x$.

6. Find (a) the derivative, and (b) the integral of the function $f(x) = x\ e^{1+x^2}$.

7. Find $\int (x^2+5)\ \cos x \, \mathrm{d}x$.

8. (a) i. $\int xe^{x^2}\ \mathrm{d}x$

ii. $\int xe^{2x}\ \mathrm{d}x$

iii. $\int x^2e^x\ \mathrm{d}x$

(b) $\int x(x^2+3)^5\ \mathrm{d}x$

(c) $\int \frac{3x-1}{x^3-x^2}\ \mathrm{d}x$

(d) $\int \sin\theta\cos\theta\ \mathrm{d}\theta$

(e) $\int \frac{1+\sec^2 x}{x+\tan x}\ \mathrm{d}x$

9. 'Eight integrals in 25 minutes,' and simplify your results.

(a) $\int \cos^5 x\ \sin x\ \mathrm{d}x$

(b) $\int 2x\ \sin 2x\ \mathrm{d}x$

(c) $\int \frac{x-1}{\sqrt{x}}\ \mathrm{d}x$

(d) $\int \sec^2\theta\ e^{\tan\theta}\ \mathrm{d}\theta$

(e) $\int \frac{18}{x^2+9}\ \mathrm{d}x$

(f) $\int \frac{18}{x^2-9}\ \mathrm{d}x$

(g) $\int \frac{18x}{x^2+9}\ \mathrm{d}x$

(h) $\int_5^{12} \frac{2}{2x-3}\ \mathrm{d}x$

10. Perform the following integrations:

(a) $\int x^2(2x-3)\ \mathrm{d}x$

(b) $\int \frac{2}{\sqrt{1-9x^2}}\ \mathrm{d}x$

(c) $\int xe^{2x-3}\ \mathrm{d}x$

(d) $\int \cot x\ \mathrm{d}x$

(e) $\int \cos^2 x\ \mathrm{d}x$

11. Integrate the following functions: (a) xe^{x^2}, (b) xe^{2x} .

12. Find the integral $\int \frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x$ by using (a) the substitution $x=\sin\theta$, (b) the substitution $x=\cos\theta$.

Explain why the results of the integrations look different. (It may help to sketch the two integral functions on the calculator.)

13. The gradient function of $y = f(x)$ is $f'(x) = e^{2x} + 1$, and the graph of $y = f(x)$ passes through the point $P(0.5, 1.5)$. Find the function $f(x)$.

14. Evaluate the following definite integrals precisely:

(a) $\displaystyle\int_0^1 \frac{4x}{\sqrt{1-x^2}}\,\mathrm{d}x$

(b) $\displaystyle\int_{-1}^1 \frac{2}{4-x^2}\,\mathrm{d}x$

15. (a) Do $\displaystyle\int x\ln x\,\mathrm{d}x$.

(b) If $\displaystyle\int_0^{\pi/2} x\,\cos x\,\mathrm{d}x = p\pi + q$, where $p, q \in \mathbb{Q}$, find p and q.

16. In this problem, let $C_n = \displaystyle\int_0^{\pi/2} \cos^n x\,\mathrm{d}x$. Evaluate C_1 .

Given the reduction formula $C_n = \dfrac{n-1}{n}\,C_{n-2}$, find the value of $\displaystyle\int_0^{\pi/2} \cos^7 x\,\mathrm{d}x$, giving your answer precisely, as a fraction reduced to its lowest terms. (Note: you are not asked to prove that reduction formula.)

17. It is given that the equation of motion of a body moving under gravity is $\dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -g$, where you should take $g = 10\,\mathrm{m/sec}^2$, and s and t denote the vertical displacement of the body and time, respectively.

At the Olympics in Barcelona, a diver jumped upwards at time $t = 0$ sec with a velocity of $v = 6$ m/sec from a springboard which is $s = 8$ m above the water surface.

(a) By integrating the equation of motion and determining the value of the constant of integration, find an expression for the velocity $v(t)$ at time t.

(b) By integrating your expression for $v(t)$ and determining the constant of integration, find an expression for the height $s(t)$ of the diver above the surface at time t.

(c) Hence calculate the time after his jumping up when he hit the surface of the water, *i.e.* when $s = 0$, and calculate his velocity at that moment.

18. A car is standing still next to me on a long straight road. After one minute (at $t = 1$ min) it starts to move, and for the next three minutes its speed in km/min is given by $v = \frac{1}{27}(-4t^3 + 30t^2 - 48t + 22)$. After that, from $t = 4$ min onwards, it continues to move along at the speed it has reached.

(a) Calculate the speed of the car at $t = 1$ min and $t = 4$ min. Use your calculator to sketch a diagram of the speed $v(t)$ of the car between $t = 0$ min and $t = 7$ min and describe the movement of the car in a brief sentence.

(b) Find a formula for the acceleration $a(t)$ of the car for the time between $t = 1\,\text{min}$ and $t = 4\,\text{min}$, and sketch its graph.

(c) At what time does the car accelerate most strongly and what is its greatest acceleration, in km/min^2 ?

(d) What distance does the car travel between $t = 4\,\text{min}$ and $t = 7\,\text{min}$?

(e) Similarly, what distance does it travel between $t = 1\,\text{min}$ and $t = 4\,\text{min}$?

(f) How far then has the car travelled from where I am still on the side of the road in the first 7 minutes, and what has been its average speed during that time?

Areas and Volumes

19. (a) Evaluate the following definite integral: $\displaystyle\int_1^{13} \frac{1}{2x+1}\,\mathrm{d}x$, giving your answer in the form $\ln p$, where $p \in \mathbb{N}$.

(b) Find the area below the graph of $f(x) = 2\sin x$, between $x = 0$ and $x = \frac{1}{2}\pi$. Hence find the area bounded by the graph of that function, the y-axis and the line $y = 2$.

20. Find one of the (infinitely many) areas enclosed by the x-axis and the graph of $y = 1 - \cos x$. (It may help to sketch the graph first ...)

21. Find the coordinates of the points of intersection of the graphs of $y = x^2+2$ and $y = x+8$.

Then find the area enclosed by the two graphs.

22. Find the points of intersection of the graphs of $y = \dfrac{2}{x}$ and $y = -2x + 5$, and calculate the area enclosed by the two curves.

23. Consider the graph of $y = \sin x$ for $0 \leq x \leq \pi$. Find the points of intersection of that graph with the line $y = 0.5$, and hence the area bounded by the curve and the line. To gain full marks, give your answer for the area in the form $p\sqrt{3} + q\pi$, where $p, q \in \mathbb{Q}$.

24. Find the area bounded by the graph of $y = x^2 + 1$, the x -axis, and the two lines $x = \pm 2$.

Hence find the area bounded by the graph of $y = x^2 + 1$ and the line $y = 5$.

25. (a) i. Evaluate the integral $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$.

ii. Use your calculator to sketch the graph of $y = \dfrac{1}{\sqrt{x}}$, and mark the area corresponding to the definite integral you calculated in part i.. What do you observe?

(b) Consider the function $f(x) = \dfrac{x(x-2)}{(x+1)(x-3)}$.

i. Find the points of intersection of the graph of $y = f(x)$ with the axes, and the vertical and horizontal asymptotes, and use this information to sketch the graph.

ii. Differentiate the function $f(x)$ and **hence** determine the coordinates of the stationary point of the curve.

iii. Integrate the function $f(x)$ and simplify your result.

iv. Calculate the small area bounded by the graph of $y = f(x)$ and the x-axis.

26. (a) Find the values of

$$\int_0^{\frac{\pi}{2}} x \sin x \, \mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{\pi}{4}} \sin^3 x \cos x \, \mathrm{d}x$$

(b) Sketch the graphs of $y = \sin x$ and $y = \cos x$, for $0 \leq x \leq 2\pi$, in the same diagram. Find the area enclosed by the two graphs.

(c) Find the volume of revolution when the area bounded by the x-axis, the graph of $y = \sqrt{x}$ and the line with equation $x = 4$ is rotated about the x-axis.

27. Consider the function $y = \sqrt{4 - x^2}$, defined for $-2 \leq x \leq 2$.

(a) Differentiate the function. Hence find the stationary point (turning point) and determine its nature.

Find the intersections of the graph of the function with the axes, and investigate the gradient function as $x \to \pm 2$. Hence sketch the graph of the function.

(b) Use the substitution $x = 2 \sin \theta$ to evaluate the integral $\int_0^2 \sqrt{4 - x^2} \mathrm{d}x$. Deduce the area of a circle of radius 2.

(c) Find the volume generated when the area bounded by the positive x-axis and the positive y-axis and the curve is rotated around the x-axis. Deduce the volume of a sphere of radius 2.

28. (a) Sketch the graphs of the functions $y = \sin x$ and $y = \cos x$ for $-\pi \leq x \leq \pi$, and write down the coordinates of the points of intersection, giving the values precisely. Let P be the point of intersection of the two graphs with the smallest positive x-coordinate.

(b) Find the precise value of the area enclosed by the two graphs between those points of intersection in your diagram.

(c) What is the equation of the tangent to the graph of $y = \sin x$ at P ?

(d) Find the angle that the tangent you found in part (c) makes with the x-axis, and hence find the angle at which the graphs of the two functions intersect at P, accurate to 3 significant figures.

29. Sketch the graph of the function $y = \dfrac{1}{x^2}$ for $x > 0$ and find, in terms of k, the area below the graph between $x = 2$ and $x = k$, where $k > 2$.

Discuss what happens to that area as $k \to \infty$ (i.e. as k increases more and more,) and deduce the area below the graph to the right of $x = 2$.

30. (a) Given the function $f(x) = e^{-x^2/2}$, find its derivative $f'(x)$ and hence find the coordinates of any stationary point(s) the graph of the function may have.

(b) Find the second derivative $f''(x)$, and use it to determine the nature of the stationary point(s) you found in part (a), and to find the coordinates of any points of inflection of the graph of $y = f(x)$.

(c) Hence sketch the graph of $y = f(x)$, labelling clearly any points you previously found, and any intersection(s) with the axes.
Write down the equation of the asymptote of the graph.

(d) It is given that as $N \to \infty$, the integral $\int_{-N}^{N} e^{-x^2/2}\,\mathrm{d}x \to L$, (i.e. as N becomes larger and larger, the value of the integral comes closer and closer to some fixed number L.) Use your calculator, and a suitable (not too large) number N, to find the value of L accurate to 3 decimal places.
L can actually be written in the form $\sqrt{k\pi}$, where $k \in \mathbb{N}$. From your approximation, find k.

(e) Using a suitable substitution, find the indefinite integral $\int xe^{-x^2/2}\,\mathrm{d}x$. Hence find, in terms of N, the value of the definite integral $M = \int_{-N}^{N} xe^{-x^2/2}\,\mathrm{d}x$. What happens to the value of M as $N \to \infty$?

(f) It is also given that as $N \to \infty$, the integral $\int_{-N}^{N} x^2e^{-x^2/2}\,\mathrm{d}x \to V$. Find the value of V accurate to 3 decimal places. What then is the value of $\frac{V}{L}$?

(g) This problem has been about differentiation and integration, but what is the significance of the function that has been discussed?

Differential Equations

31. (a) A car is rolling freely to a stop, slowed down by friction only, so that the deceleration (*i.e.* negative acceleration, acceleration being the rate of change of velocity,) is proportional to the square root of the velocity. Set up a differential equation for the velocity v and find its general solution.

(b) If the initial velocity of the car was 100 km/h and the velocity after 1 minute (*i.e.* at time $t = \frac{1}{60}$ h) is 16 km/h, show that the velocity at time t, measured in hours, is $v = 100(1 - 36t)^2$. After how much time will the car come to a stop?

(c) Using the fact that $v = \mathrm{d}s/\mathrm{d}t$, write down a differential equation for the displacement s and solve it. How far will the car have travelled before it comes to a stop?

[Over most of the velocity range in the problem, friction would actually be approximately proportional to the square of the velocity, not to its square root.]

32. Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{y}$.
Sketch the graphs of the solution for three different values of the constant.
Find the equations of the two curves of the 'family' that pass through the point $(1, 4)$.

33. Find the general solutions of the following differential equations:

(a) $x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 1.$

(b) $x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y.$

34. Solve the differential equation $(x^2 - 5x + 6)\frac{\mathrm{d}y}{\mathrm{d}x} = y$, giving your answer in the form $y = f(x)$, and find the particular integral such that $y = 3$ when $x = 4$.

Numerical Integration

35. (a) Evaluate the definite integral $\int_1^3 \sqrt{x}\,\mathrm{d}x$ precisely.

(b) Use the trapezium rule with 5 ordinates, (*i.e.* with 4 strips,) to find an approximation to the value of the integral in (a).

(c) From the results of (a) and (b) deduce an approximation to the value of $\sqrt{3}$. What is the percentage error of your estimate?

36. (a) List the values of the function $f(x) = \sqrt{1 - x^2}$ for $x = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$.

(b) Find the derivative $f'(x)$ of the function and evaluate it for $x = 0$ and $x = 1$.

(c) Use your answers to the previous two parts to sketch the graph of $y = f(x)$ for $0 \le x \le 1$. What simple curve is the graph a part of?

(d) Use the trapezium rule with 6 ordinates, *i.e.* with 5 strips, to find the area under the graph approximately.

(e) Use your answers to the last two parts to estimate the value of π .

10 Vectors

1. A swimmer in heading due West at a speed of 2 km/h while being carried along by a stream flowing North to South, also at 2 km/h. What are the speed and direction of the swimmer relative to land?

2. A plane flying at 500 km/h is heading on a bearing of 210° in a storm blowing at 200 km/h from due West. At what speed is the plane moving relative to the ground, and on what bearing?
(The bearing of a direction is the angle it makes with the north, measured clockwise, *i.e.* towards the east.
You are reminded that in a triangle $c^2 = a^2 + b^2 - 2ab\cos C$ and $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.)

3. From his overnight-camp, a hiker has walked 10 km to the south-east, and then 5 km on a bearing of 75°, but he can still see the camp. How far away is the camp now, and on what bearing from his present position?
(The bearing of a direction is the angle it makes with the north, measured clockwise, *i.e.* towards the east.
You are reminded that in a triangle $c^2 = a^2 + b^2 - 2ab\cos C$ and $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.)

4. (a) The vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ form edges of a rectangular box. You are sitting inside this box, in such a way that $\overrightarrow{OA}$ is running from left to right in front of you and $\overrightarrow{OB}$ is pointing upwards.
If $\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{OB}$ and $\vec{c} = \overrightarrow{OC}$, express the following vectors in terms of $\vec{a}, \vec{b}$ and $\vec{c}$:

    i. from O to the upper right-hand corner in front of you;
    ii. from O to the upper corner behind you on the right;
    iii. from the lower corner behind you on the left to A;
    iv. from B to the lower corner behind you on the right.

   (b) P, Q, R and S are any four points. Simplify the following vector expressions:

    i. $\overrightarrow{PQ} - \overrightarrow{PR}$
    ii. $\overrightarrow{QR} - \overrightarrow{SR} + 2\,\overrightarrow{SQ}$.

5. (a) If $\vec{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$, solve the equation $3\vec{a} - 2\vec{x} = \vec{b}$.

   (b) For what values of k are the vectors $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} k \\ -3 \end{pmatrix}$

    i. parallel,
    ii. perpendicular?

6. (a) Given that $\vec{p} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, find the values of $\lambda \in I\!R$ such that $\vec{p} + \lambda\vec{q}$ is

    i. parallel to the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$,

ii. perpendicular to the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

(b) Find any vector $\vec{x}$ perpendicular to $\vec{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Then find the unit vector $\widehat{\vec{x}}$, *i.e.* the vector of magnitude 1 in the direction of $\vec{x}$.

7. (a) Find the cosine of the angle between $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Estimate what the angle is by drawing a diagram, using a unit of $1 = 2\,\text{cm}$.

(b) Write down the unit vector in the same direcion as $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$.

8. (a) Find the values of k, l and m such that $\begin{pmatrix} k \\ -5 \\ 2 \end{pmatrix} + \begin{pmatrix} -7 \\ l \\ 1 \end{pmatrix} + m\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(b) Find the angle between the vectors $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$.

9. Consider the vectors $\vec{p} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} 3 \\ k \\ l \end{pmatrix}$.

(a) If $l = -1$, find the value of k so that $\vec{p}$ and $\vec{q}$ are perpendicular.

(b) Find the values of k and l so that $\vec{p}$ and $\vec{q}$ are parallel.

(c) If $k = 4$ and $l = 0$, what is the angle between the vectors?

10. It is given that $\vec{r} = -\vec{\imath} + 3\vec{\jmath}$ and $\vec{s} = 4\vec{\imath} + \vec{\jmath}$. Find

(a) the magnitude of the resultant $\vec{r} + \vec{s}$, and

(b) the angle between the vectors $\vec{r}$ and $\vec{s}$.

(c) Represent the four vectors $\vec{r}$, $\vec{s}$, $\vec{r} + \vec{s}$ and $\vec{r} - \vec{s}$ in a single diagram.

11. A rectangular room has a floor area of 5 m by 4 m, and a height of 2.5 m. Choosing suitable base-vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$, represent a diagonal across the floor, and the diagonal from the same corner of the floor to the opposite corner on the ceiling in vector form. Hence, or otherwise, find the angle between the two diagonals.

12. Do not use a calculator in this problem. It may help to draw simple sketch in two dimensions.

The triangle ABC has vertices at $A(1,7,2)$, $B(0,4,-3)$ and $C(4,2,1)$.

(a) Find the coordinates of the midpoint M of side BC.

(b) What are the lengths of the side BC and the median AM?

(c) Calculate the value of the scalar product $\overrightarrow{BC} \cdot \overrightarrow{AM}$. What kind of triangle is ABC therefore?

(d) Use the results of parts (b) and (c) to find the area of the triangle ABC.

13. (a) Given the three points $P(1,-2,3)$, $Q(1,3,-4)$ and $R(0,1,-1)$, find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and calculate their vector-product $\overrightarrow{PQ} \times \overrightarrow{PR}$.

Hence find the area of the triangle PQR.

(b) By considering the 2-dimensional xy-plane as the plane in 3-dimensional space where $z = 0$, adapt the method used in part (a) to find the area of the triangle in the xy-plane with vertices $X(2,-1)$, $Y(3,4)$ and $Z(-2,2)$.

14. A triangle has vertices $P(-1,5,2)$, $Q(2,5,6)$ and $R(1,4,0)$.

(a) Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and their magnitudes, and use the scalar product to find the cosine of the angle between them. Hence find the area of the triangle.

(b) Also calculate the vector product $\overrightarrow{PQ} \times \overrightarrow{PR}$, and use it to find the area of the triangle again, showing clearly how you arrive at your answer.

15. The points A and B are the ends of a diameter of a circle of radius r, and C is any other point on the circumference.
Using the centre O of the circle as the origin, express $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of the position vectors of A and C (only), and hence show that the angle subtended by AB at C is a right angle.

16. From O-town, there is a straight road to A-ville, which is 9 km to the East and 6 km North of O-town. Mount B is 5 km West and 1 km South of A-ville. At the moment there is only a dirt road to Mount B, which runs straight from O-town.

(a) Choose a suitable vector basis, and express the vector from O-town to Mount B in terms of that basis. How long is that road?

As tourism to Mount B has increased greatly, it is decided to build a straight road from Mount B to join with the road from O-town to A-ville at some point X, but to save costs, the new road should be as short as possible.

(b) Suppose that X lies a fraction λ along the way from O-town to A-ville, and express the road from Mount B to X as a vector, in terms of λ.

(c) Use the condition that the new road should be as short as possible to determine the value of λ. Hence describe the location of X relative to O-town.

3-Dimensional Vector Geometry

17. Rewrite the following Cartesian equations of three lines in parametric form, and find the angle between the <u>first two</u> lines:

$$\frac{x-2}{2} = \frac{y+3}{-1} = \frac{z}{2}, \qquad \frac{x-1}{6} = \frac{y-1}{2} = \frac{1-z}{3}, \qquad \frac{x-7}{3} = \frac{z+11}{5}, y = 2$$

18. It is given that of the following three lines

$$\ell_1 : \vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad \ell_2 : \vec{r} = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}, \quad \ell_3 : \vec{r} = \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

two are parallel, one is perpendicular to the other two, and two intersect in one point. State, with reasons, which ones are parallel and which perpendicular, and find the coordinates of the point of intersection of the two lines that intersect.

19. (a) Convert the following equation of a line ℓ to vector (parametric) form:
$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{2z}{3} .$$
(b) Convert the following equation of a plane Π to Cartesian form:
$$\vec{r} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} .$$
(c) Hence find the angle between the line ℓ and the plane Π .

20. Consider the planes $\Pi_1 : x - 2y + 2z = 2$ and $\Pi_2 : 3x - 4z = 12$.

(a) Calculate the cosine of the angle between the two planes;

(b) find the vector-product of the normals of Π_1 and Π_2, and

(c) the coordinates of <u>any</u> point that lies on both planes, and hence

(d) a vector equation for their line of intersection; and

(e) obtain the point of intersection of the two planes with a third plane, $\Pi_3 : x - y + z = -1$.

21. (a) Determine if the following planes are parallel.

i. If they are, find the distance between them;

ii. if they are not, find the angle between them and the equation of their line of intersection.

$$x + y - z = 3 \quad \text{and} \quad 2x - y + 2z = -4 .$$

(b) Determine if the following line and plane are parallel.

i. If they are, find the distance between them;

ii. if they are not, find their point of intersection.

$$\vec{r} = \begin{pmatrix} -1 \\ -11 \\ 7 \end{pmatrix} + \kappa \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{r} \cdot \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix} = -9 .$$

22. Given the plane with normal equation $\vec{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = -1$, and the point $P(6, 3, -1)$,

(a) find the vector equation of the line ℓ through the point P and perpendicular to the plane, and the coordinates of the point F of intersection of ℓ and the plane.

(b) Hence find the distance of point P from the plane, and

(c) the coordinates of the image point P' when the point P is reflected in the plane.

23. Find the equation of the line through $P(-1, 2, 4)$ perpendicular to the plane with equation $\vec{r} \cdot \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = 26$.
Then find the point F of intersection of the plane and the line, and hence the distance of P from the plane.
Also find the point P' which is the reflection of point P in the given plane.
(Note that all coordinates and other values in this problem are integers.)

24. (a) Find the equation of the plane through the three points $K(3, 2, -1)$, $L(4, -2, 3)$ and $M(3, -2, 0)$, in vector (or parametric) form.

(b) Deduce that the Cartesian form of the equation of that plane is $12x - y - 4z = 38$.

(c) Find the cosine of the angle between this plane and a second plane with equation $3x + y - z = 7$, and also the equation of the line of intersection between the two planes.

(d) Find the point of intersection of the two planes with a third plane, $x + 2y = 0$.

(Note that the last two parts of the problem can be done without having done the first two parts.)

25. Find the area of the triangle with vertices $A(1, 1, 0)$, $B(-3, 2, 0)$, $C(0, -3, 0)$.
What is the distance of the point $D(-2, 0, 4)$ from the plane of the triangle? (This question does not require any calculation.)
Hence, or otherwise, find the volume of the tetrahedron $ABCD$.

11 Complex Numbers

1. (a) For which z is $(4+i)z + (-11+10i) = 0$?

   (b) Solve the following equations simultaneously:
   $$2i\ z + (1+i)\ w = 2+2i,$$
   $$3\ z + (2-i)\ w = 5-i.$$

   (c) Find the two values of z such that:
   $z^2 + (-2-2i)z + (-4+2i) = 0.$

2. (a) Use the algebraic method to find the square roots of $z = 12 + 16\,i$, and represent them in an Argand diagram.

   (b) Use the standard formula, and the result of part (a) above, to solve the following quadratic equation: $i\,z^2 + (2-2\,i)\,z + (-6+3\,i) = 0$.

3. (a) Find a quadratic equation with real coefficients, one of whose roots is $1-2i$.

   (b) Find a quadratic equation whose roots are $1-2i$ and $2+i$.

   (c) Solve the quadratic equation $iz^2 + (2+4i)z + (4-13i) = 0$.

4. (a) If $z = -3 + 4\,i$, express z, its complex conjugate $\bar{z}$ and its reciprocal $\frac{1}{z}$ in modulus-argument form.

   (b) A quadratic equation with real coefficients has $(1 - \frac{1}{2}\,i)$ as one of its roots (or solutions.) Write down the other root, and find the simplest such equation with all integer coefficients.

5. (a) Solve the following equation for z: $(3-4\,i)\,z - 2 + i = 0$.

   (b) Find the moduli and arguments of $z = -2-2\,i$ and $w = 1+\sqrt{3}\,i$. Hence calculate the modulus and argument of (a) zw , (b) $\dfrac{z}{w}$.

6. (a) Use the algebraic method to find the square roots of $z = -24 - 10\,i$, and represent them in an Argand diagram.

   (b) Use deMoivre's Theorem to find the modulus and arguments of the cube roots of $z = -4\sqrt{2} + 4\sqrt{2}\,i$, and represent the roots in an Argand diagram.

7. Solve the following equations by the methods specified. (Pay attention to the number of solutions in each case.)

   (a) $(1-2i)\ z + (2-i) = -1+i$, by subtractiung and dividing;

   (b) $z^2 = 15 + 8i$, by setting up and solving simultaneous equations for the real and imaginary parts of z ;

   (c) $z^2 = 1 - 2i$, using your calculator, giving the real and imaginary parts accurate to 2 decimal places;

(d) $z^3 + 8 = 0$,

i. by rewriting -8 in modulus-argument form and using deMoivre's theorem, and

ii. by first factorising the LHS using the identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

8. (a) i. Write $3\sqrt{2}\, e^{\frac{3\pi}{4}i}$ in the form $a + bi$ where $a, b \in I\!R$.

ii. Express $-5 + 12i$ in modulus-argument form.

(b) Using the algebraic method, find the square roots of $z = -8 - 6i$ and represent z and its two roots in an Argand diagram.

(c) Use deMoivre's theorem to find the modulus and the arguments of the five 5th roots of $z = -32$ and represent these roots in an Argand diagram.

9. By rewriting $e^{e^{2+\frac{\pi}{3}i}}$ in the form $r\, e^{i\theta}$, find the precise values of the modulus and the argument of the number.

10. (a) Represent the following *locus* in an Argand diagram: $\arg(z - 1 - i) = \frac{5\pi}{4}$.

(b) Use the substitution $z = x + iy$ to express the following condition in terms of x and y and simplify it; sketch the locus in an Argand diagram: $|\, z - 1| = \Im(z + i)$.
(Note that $\Im(z)$ denotes the imaginary part of the complex number z, so that if $a, b \in I\!R$ then $\Im(a + b\, i) = b$.)

11. (a) Represent the following *locus* in an Argand diagram: $|\, z - 2 + I| = 2$.

(b) Use the substitution $z = x + iy$ to express the following condition in terms of x and y and simplify it; sketch the locus in an Argand diagram: $\Re\,(\frac{1}{z}) = \Re\,(z)$.
(Note that $\Re\,(z)$ denotes the real part of the complex number z, so that if $a, b \in I\!R$ then $\Re\,(a + b\, i) = a$.)

12. Sketch the following *loci* in separate Argand diagrams.

(a) $\arg(z - 2 - i) = -\frac{3\pi}{4}$,

(b) $|\, 1 - i - z| = 1$,

(c) $|\, z + 1| = |\, z - 1 - 2i|$,

(d) $\Re\,(\,(\bar{z} - 6)\, z\,) = -5$, where $\bar{z}$ is the complex conjugate of z. (Hint: let $z = x + iy$.)

13. (a) Mark the following *loci* in one Argand diagram.

i. $|\, z + 2 - i| = 2$,

ii. $\arg\,(z + 2 - 3i) = -\frac{\pi}{4}$,

iii. $|\, z + 1| = |\, z + 3 - 2i|$.

(b) If $z = x + iy$, write the following *locus* in Cartesian form: $\Re(z^2) + 2z\bar{z} = 3$.

12 Linear Transformations and Matrices

1. Solve the following equation to find the matrix X: $\begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 0 \end{pmatrix} - 2X = \begin{pmatrix} 5 & 5 & 0 \\ 1 & -2 & 20 \end{pmatrix}$.

2. Solve the following matrix equations to find the matrix X in each case:

(a) $2X + \begin{pmatrix} 3 & -1 & 2 \\ -4 & 2 & 0 \end{pmatrix} = 3\begin{pmatrix} 5 & 3 & 0 \\ -1 & -2 & 20 \end{pmatrix}$,

(b) $X\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$.

3. Which, if any, of the following are not valid matrix identities? State your reasons carefully.

(a) $(A+2B)^2 = A^2 + 4AB + 4B^2$,

(b) $(A+\lambda I)^3 = A^3 + 3\lambda A^2 + 3\lambda^2 A + \lambda^3$, where λ is a scalar and I is the identity matrix.

4. Solve each of the following equations for the matrix X:

(a) $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix} - 2X = \begin{pmatrix} 3 & 4 & 0 \\ -2 & 1 & -3 \end{pmatrix}$.

(b) $X\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

5. A linear transformation T has matrix $M_T = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

(a) If the triangle OAB has vertices $O(0,0), A(4,0)$ and $B(2,2)$, find the vertices of its image $OA'B'$ under the transformation T.

(b) Choosing a suitable scale, sketch the triangles OAB and $OA'B'$ in the same diagram.

(c) Calculate the area of triangle OAB and the determinant $\det M_T$ of the matrix, and hence find the area of triangle $OA'B'$.

(d) By comparing the triangle OAB and its image, try to describe the effect of the linear transformation T.

(e) Find the inverse M^{-1} of the matrix M, <u>and hence</u> solve the following simultaneous equations:

$$\begin{aligned} x - 2y &= 7 \\ 2x + y &= -9 \end{aligned}$$

6. (a) The linear transformation T transforms the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ into the vector $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ into $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Find the matrix M_T of that linear transformation.

Into what vector is $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ transformed by T ?

(b) The linear transformation with matrix $M = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ maps the triangle with vertices A, B and C to the triangle with vertices $A'(1,1)$, $B'(4,1)$ and $C'(3,5)$.

i. Find the coordinates of vertex C.

ii. Sketch the triangle $A'B'C'$ and find its area. Then use the determinant of M to deduce the area of triangle ABC.

iii. Describe the linear transformation.

7. It is given that $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(a) Find the following matrices: $AB, A^{-1}, B^{-1}, (AB)^{-1}, A^{-1}B^{-1}, B^{-1}A^{-1}$.
What do you observe? Try to generalise.

(b) Use a matrix method to solve the simultaneous equation: $\begin{aligned} 2x + y &= 1 \\ -x + 2y &= 4 \end{aligned}$

(c) What types of linear transformations do matrices A and B represent?

8. (a) i. Describe in geometrical terms the linear transformation represented by the matrix $P = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$. (Hint: Consider the effects on the vectors $\vec{i}$, $\vec{j}$ and $\vec{i}+\vec{j}$.)

ii. Write down the inverse matrix P^{-1} of P.

iii. If $Q = \begin{pmatrix} -1 & -4 \\ 4 & -1 \end{pmatrix}$, find the product PQ of the two matrices.

(b) i. Rewrite the complex number $z = 3 + 2i$ in modulus-argument form.

ii. Find the complex number $\dfrac{1}{z}$, giving your answer in the form $a + bi$.

iii. If $w = -1+4i$, find the product zw of the two numbers, again in the form $a+bi$.

(c) i. Calculate the complex number $\dfrac{-2+5i}{3-4i}$, again giving your answer in the form $a + bi$.

ii. Briefly outlining your reasoning, use the results of the previous parts of this problem to write down the matrix $\begin{pmatrix} -2 & -5 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}^{-1}$.

9. (a) Given the two matrices $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, show that the determinant of their product is the product of their determinants, *i.e.* that $\det(XY) = (\det X)(\det Y)$.

(b) Hence prove by induction that if A is any 2×2 -matrix $\det(A^n) = (\det A)^n$, for all $n \in \mathbb{N}$.

(c) If A is an $n \times n$ -matrix and $\lambda \in \mathbb{R}$ is any scalar, is the following a valid identity: $\det(\lambda A) = \lambda \det A$? If it is not, give a corrected version of it.

(d) In 3-dimensional space, with the usual base vectors $\vec{i}, \vec{j}$ and $\vec{k}$, the vector operation $\otimes$ is defined as follows: if $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, then $\vec{x} \otimes \vec{y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ p & q & r \end{vmatrix}$.

If $\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix}$,

i. evaluate the determinant $\vec{x} \otimes \vec{y}$ and write the result as a column vector,

ii. find the angles between $\vec{x} \otimes \vec{y}$ and $\vec{x}$, and between $\vec{x} \otimes \vec{y}$ and $\vec{y}$,

iii. find the cosine of the angle θ between $\vec{x}$ and $\vec{y}$, and verify that $|\vec{x} \otimes \vec{y}| = |\vec{x}|\,|\vec{y}| \sin\theta$.

What other vector operation do these results suggest that the operation $\otimes$ is the same as?

10. Find the value of k so that the matrices $\frac{1}{21}\begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ k & 0 & 7 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$ are inverses of each other. Hence solve the following system of equations:

$$\begin{aligned} x - 2y &= 0 \\ 3x + y + 5z &= 2 \\ -x + 2y + 3z &= 1. \end{aligned}$$

11. Consider the following system of equations:

$$\begin{aligned} x + 2y + z &= a \\ 2x + y + z &= 0 \\ \lambda x - y - 2z &= 0 \end{aligned}$$

(a) If $\lambda = 5$ and $a = 7$, rewrite the set of equations in the form $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{v}$, use your calculator to find the inverse matrix M^{-1}, and hence find the solution of the equations.

(b) If $a = 0$, this is a set of homogeneous equations. For which value of λ do the equations have a non-trivial solution, i.e. a solution other than $x = y = z = 0$? Write down one such other solution.

12. For which value of p does the following system of homogeneous equations have a non-trivial solution? For that value of p, find the general form of the solution.

$$\begin{aligned} x - y + z &= 0 \\ 2x - y - z &= 0 \\ px + 6y + 2z &= 0. \end{aligned}$$

13. A farmer is moving from crops to lifestock, by investing in land and buildings, and buying dairy cattle, pigs and horses; of course the different kinds of animals have different requirements of land area and building space, as shown in the first matrix. He has to decide between two plans, which are as shown in the second matrix.

$$R = \begin{array}{r} \\ \textit{land} \\ \textit{building} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{cattle} & \textit{pigs} & \textit{horses} \end{array} \\ \begin{pmatrix} 10 & 3 & 6 \\ 4 & 7 & 9 \end{pmatrix} \end{array} \qquad P = \begin{array}{r} \\ \textit{plan I} \\ \textit{plan II} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{cattle} & \textit{pigs} & \textit{horses} \end{array} \\ \begin{pmatrix} 20 & 40 & 10 \\ 40 & 20 & 0 \end{pmatrix} \end{array}.$$

(a) Consider the four 'products' RP, R^TP, RP^T and R^TP^T, (where M^T denotes the transpose of the matrix M.) Which of these four is, or are, defined? Which of them is meaningful?

(b) Calculate the meaningful one of the four 'products' in (a). Explain what kind of information this product gives.

(c) The cost of the land used is £15/unit and the cost of the building space is £20/unit. Represent this information in another matrix, and use it to calculate the costs of the two plans.

14. The results from a survey in two apartment blocks, A and B, of the numbers of male and female residents (humans) and of male and female pets (dogs and cats) are summarised in the matrices M and F; matrix Q summarises some well-known facts of biology.

$$M = \begin{array}{r} \\ \text{A} \\ \text{B} \end{array} \begin{array}{c} \begin{array}{cc} \text{mh} & \text{mp} \end{array} \\ \begin{pmatrix} 8 & 3 \\ 11 & 2 \end{pmatrix} \end{array} \qquad F = \begin{array}{r} \\ \text{A} \\ \text{B} \end{array} \begin{array}{c} \begin{array}{cc} \text{f h} & \text{f p} \end{array} \\ \begin{pmatrix} 9 & 4 \\ 10 & 5 \end{pmatrix} \end{array} \qquad Q = \begin{array}{r} \\ \text{humans} \\ \text{pets} \end{array} \begin{array}{c} \begin{array}{ccc} \text{heads} & \text{arms} & \text{legs} \end{array} \\ \begin{pmatrix} 1 & 2 & \\ & & 4 \end{pmatrix} \end{array}$$

(a) Copy and complete the matrix Q in the most suitable way.

(b) Decide which of the following matrix expressions (i) can be calculated and (ii) are meaningful. Calculate the ones that are meaningful, and very briefly describe the information they show.

$MF \qquad M+Q \qquad MQ \qquad QF \qquad M+F \qquad (M+F)Q$

13 Mathematical Induction

1. Of the following two propositions only one is true. By trying different values for n, decide which is the true one and prove it by induction.

   (a) $n^2 + 2n$ is a multiple of 3 for all $n \in \mathbb{N}$.

   (b) $n^3 + 2n$ is a multiple of 3 for all $n \in \mathbb{N}$.

2. Of the following two propositions only one is true. By trying different values for n, decide which is the true one and prove it by induction.

   (a) $\sum_{r=0}^{n} 2^r = n^2 + n + 1, \quad \text{for } n = 0, 1, 2, ...$;

   (b) $\sum_{r=0}^{n} 2^r = 2^{n+1} - 1, \quad \text{for } n = 0, 1, 2, ...$.

3. (a) Prove by induction that the number of ways in which n people can be arranged around a round table is $(n-1)!$. (At a round table, if everyone just moves one seat to the right, it is the same arrangement.)

   (b) Prove by induction that $1 + 2 + 3 + ... + n = \frac{1}{2}n(n+1)$.
   From the formula for the sum, deduce the value of the sum $1 + 2 + 3 + ... + 3n$, and the value of the sum $3 + 6 + 9 + ... + 3n$, both in terms of n.
   Hence find the value of $1 + 2 + 4 + 5 + 7 + 8 + ... + 50$.

14 Probability

1. Write down how many distinct, equally likely outcomes there are when

(a) three fair dice are rolled,

(b) four people arrive at a meeting in random order,

(c) two pencils are picked at random from five pencils of different colours?

2. Write down the probabilities of the following events:

(a) that three coins that are tossed show either all Heads or all Tails;

(b) that two dice that are rolled have a total score of 10 or less;

(c) that four sheets of paper numbered 2, 4, 7, 11, picked up at random, are neither in ascending nor in descending numerical order.

3. 6 grey and 10 blue socks are hanging on a washing line, and two socks are taken at random. Write down the probabilities of the following events:

(a) the first sock is blue.

(b) both socks are blue.

(c) the socks are of the same colour.

(d) one sock is blue and the other is grey.

(e) one of the socks is red.

4. Two dice are rolled. Write down the probabilites of the following events:

(a) the second die shows a Four.

(b) both dice show a Three.

(c) the two dice show different numbers.

(d) the sum of the numbers is 4.

(e) the sum of the numbers is at most 11.

5. (a) Five coins have been tossed. What is the probability that they are neither all Heads nor all Tails?

(b) Someone has dropped five pages of an essay, numbered 1, 2, 3, 4 and 5, and picked them up in random order. What is the probability that they are in the right order or in the opposite to the right order?

6. Two coins have been tossed. If you have been told that at least one of the coins has come up Heads, what is the probability that both coins have come up Heads?

7. Six students regularly study together, and each day they decide randomly who has to buy some chocolate. In the five days from Monday to Friday, what are the probabilities that

   (a) it is always the same person who has to buy the chocolate, and

   (b) it is a different person on each of the five days?

8. (a) A combination lock has four dials, each with numbers 0, 1, 2, ..., 9. How many possible four-digit combinations are there?

   (b) Another lock has buttons labelled 0, 1, 2, ..., 9, X, Y, Z, C. Opening the lock requires pressing the C-button, followed by four different number-buttons, in any order, followed by one of the other letter-buttons (i.e. not 'C'.) How many combinations are there?

   (c) 5 different keys, for 5 similar locks, have been put back on their 5 hooks in random order.

      i. How many different arrangements are there of the keys?

      ii. If two of the keys are a little thicker, what is the probability that these two keys are hanging next to eachother?

      iii. If someone tries all the keys on the first lock until it opens, and then the remaining ones on the next lock, and so on, and it takes 4 seconds to try a key in a lock, how long might it take someone to open all the locks if they are very unlucky?

9. (a) i. Three dice are rolled. What is the probability that they all show a Six?

      ii. What is the probability that two show a Six each and one an odd number?

   (b) i. Two cards are drawn from a deck (without replacement.) What is the probability that both are Hearts?

      ii. What is the probability that one is Hearts and the other Spades?

10. A conscientious person puts eight bottles, three of white glass and five of green glass, into a bottle bank, in random order, for recycling. What are the probabilities that

   (a) the first two bottles are green,

   (b) the first bottle is green and the second bottle is white,

   (c) the first two bottles are of different colours,

   (d) the first bottle is green or the second bottle is white,

   (e) the last two bottles are both green?

11. 25% of the chips produced by a machine are faulty.

   (a) If two chips are chosen at random, what is the probability that neither is faulty?

(b) A box contains eight chips, of which two are faulty. If two chips are chosen from the box at random, what is the probability that neither is faulty?

12. Two events A and B have probabilities $p(A) = \frac{2}{3}$ and $p(B) = \frac{1}{4}$. What is $p(A \vee B)$, or $p(A \cup B)$, in the following two cases:

(a) the events A and B are mutually exclusive?

(b) the events A and B are independent?

13. If A and B are events such that $p(A) = 0.2$ and $p(B) = 0.5$, find $p(A \vee B)$ given that

(a) A and B are mutually exclusive events, and

(b) A and B are independent events.

14. A, B and C are events with probabilities $1/2$, $2/3$ and $2/5$ respectively.

(a) If A and B are independent, what is the probability of $A \vee B$?

(b) If A and C are mutually exclusive, what is the probability of $A \vee C$?

(c) If A and B are exhaustive, what is the probability of $A \wedge B$?

15. Suppose that A and B are events such that $p(A) = 2/3$ and $p(A \wedge B) = 2/5$.

(a) If A and B are independent events, find $p(B)$ and hence $p(A \vee B)$.

(b) What is $p(B)$ if A and B are exhaustive events?

What does it mean, in terms of probabilites, to say that two events X and Y are mutually exclusive?

16. The events K and L have probabilities $p(K) = 0.6$ and $p(L) = 0.3$. Find the following probabilities:

(a) $p(K \cup L)$, if K and L are mutually exclusive.

(b) $p(K \cup L)$, if K and L are independent events.

(c) $p((K \cup L)')$ (i.e. the probability of the event: not K or L), again if K and L are independent.

17. Given that $p(A) = \dfrac{3}{4}$, $p(B) = \dfrac{5}{7}$ and $p(A \vee B) = \dfrac{13}{14}$, find $p(A \wedge B)$. Check whether the events A and B are independent.

18. Given that $p(M) = 3/4$, $p(M \wedge N) = 1/2$ and $p(M \vee N) = 11/12$, find $p(N)$.

Are events M and N independent? State your reason briefly.

19. Of two coins, one is fair while on the other Heads has a probability of $\frac{2}{3}$. One of the coins is picked at random and rolled. Find the probabilities of the following events:

(a) the fair coin was picked and it came up Heads;

(b) the coin that was picked came up Heads;

(c) the fair coin was picked, given that it came up Heads.

20. Of two bags, bag X contains 3 red, 3 blue and 6 green balls, bag Y contains 6 red and 2 blue balls. One of the bags is chosen at random, and then one ball from that bag taken at random.

(a) Given that the ball is green, what is the probability that it came from bag X?

At the bottom of this page, clearly draw and label a probability tree to represent the situation, and hence find the probability that

(b) the ball is from bag Y and is red,

(c) the ball is red,

(d) the ball is from bag Y, given that it is red.

Finally, the balls in the two bags are put together and one is taken at random.

(e) What is the probability that it is a red ball?

21. According to an opinion poll in a certain society, 60% of the population are in favour of the death penalty and 20% support Amnesty International. What is the probability that a person chosen at random from the society is in favour of the death penalty <u>or</u> supports Amnesty International (or does both) in the following three cases:

(a) the two opinions are mutually exclusive;

(b) the two opinions are independent;

(c) 30% of those who support Amnesty are in favour of the death penalty?

In each case, what is the probability that the person does neither?

22. In a certain community, X is the 'event' that someone has not written a letter for AI, and Y the 'event' that (s)he is against the death penalty, where X and Y are such that $p\,(X) = 0.5$ and $p\,(Y) = 0.8$;

(a) what is $p\,(X \wedge Y)$ if X and Y are exhaustive, and

(b) what is $p\,(X \vee Y)$ if X and Y are independent?

If X and Y are exhaustive, what cannot happen in that community?

23. In a certain community, the probability of a person claiming to be environmentally conscious is $\frac{5}{6}$, and the probability that someone who claims to be environmentally conscious uses recycled paper is $\frac{3}{4}$. What is the probability that a person chosen at random is

environmentally conscious and uses recycled paper?
If the proportion of people actually using recycled paper is $\frac{2}{3}$, what is the probability that someone will claim to be environmentally conscious, given that they use recycled paper?

24. In some population, 0.2% of all people have a rare medical condition C. This condition is not noticeable, and one has to test for it, but the test is only 90% accurate, i.e. in 10% of the cases where someone does in fact have C the test gives a negative result, and in 10% of the cases where a person does not have C the test still comes out positive.

(a) Draw a probability tree to represent the situation; (start with the person either having or not having C, and in each case distinguish the test coming out positive or negative.)

(b) If a person is chosen at random, what is the probability the he or she has C and the test comes out positive?

(c) What is the probability that the test comes out positive?

(d) If my test has come out positive, what is the probability that I do in fact have condition C? (– a surprising outcome?)

25. 0.1% of the population have a certain medical condition. There is a test for that condition, but it is wrong in 20% of all cases, (i.e. 20% of all who really have the condition test negative, and 20% of those who don't have it nevertheless test positive.) Represent this information in a clearly-labelled, complete tree diagram.
For what percentage of the whole population does the test come out positive? If someone is told that they tested positive, find the probability that they actually have the condition. (The vast majority of doctors apparently think that it is about 80%. It isn't.)

26. After working in the Quiet Room of their House, three students walk out together, leaving no-one behind. Each has a probability of $\frac{1}{4}$ of thinking of switching the lights off, (and their actions are independent.) What are the probabilities (a) that no-one remembers, and hence (b) that at least one of them remembers?
How many such students would have to walk out together for the probability that at least one of them remembers to be greater than 95% ?

27. (a) Find the term without x in the expansion of $(2x^2 + \frac{1}{x})^9$.

(b) Write down the first three terms in the expansion of $(2 - 3x)^5$, starting with the lowest power of x. Substitute a suitable value for x to find an approximation for 1.97^5. Using your calculator, determine the precentage error of your approximation.

(c) Suppose birthdays are evenly distributed over all 12 months. In a group of 21 randomly selected people, how many would you expect to have a birthday this month? But out of those 21, what is the most likely number of people to have a birthday this month?

28. Five colleagues from work try car-pooling: *i.e.* they travel together to work in one car. The person who is driving his car will certainly turn up at the meeting point, but the probability for each of the others of making it is 0.9 , (and their behaviour is independent.) What is the probability that there are four people, (*i.e.* the driver and <u>three</u> others,) in the car on a given day?

29. A tetrahedral (i.e. four-sided) die, with its sides labelled Blue, Green, Red and Yellow, is rolled 5 times. What are the probabilities of

(a) Red being at the bottom (precisely) 4 times, and

(b) Red being at the bottom at most 3 times?

30. The probability that a smoker will die early, of a smoking-related disease, is about 40%. Using that figure, find the probabilities, as percentages, that of four smoking friends

(a) no-one will die of a smoking-related disease,

(b) just one of them will, and

(c) at least two of them will die of smoking-related diseases.

Random Variables

31. What is the expected value of the following game for me?
A fair coin is tossed, and if it comes up Heads I have to pay you £2 and the game is over; but if it comes up Tails, it is tossed again. If on this second toss it comes up Heads, I have to pay you £1 and the game is over; but if it comes up Tails again, you have to pay me £6.

32. According to a certain model, when someone has contracted the disease D at time $t = 0$, the p.d.f. of the onset of symptoms is as follows:

$$\phi(t) = \begin{cases} 0 & : \quad t < 5 \\ k(t-5)(9-t) & : \quad 5 \le t < 9 \\ 0 & : \quad t \ge 9 \end{cases}$$

where t is the number of days.
Determine the value of k, sketch the graph of the p.d.f., and write down the mean time until the onset of symptoms.
On this model, after how much time will ⅓ of those infected have started to show the symptoms?

15 Statistics

Descriptive Statistics

1. When the ages of twenty children were added up, the sum was found to be 180; when the squares of their ages were added up, the sum was found to be 1700. Using the definition of the mean, and the formula below for the variance (with a suitable value for A), find the mean and the standard deviation of the ages of the children in the group.

   Formula for the variance $\quad s^2 = \dfrac{\sum (x_r - A)^2 f_r}{\sum f_r} - (m - A)^2.$

2. The number of letters in the words of a brief abstract of a book were counted, and the following distribution of word-lengths was found:

letters/word:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
word frequency:	41	130	114	114	91	64	59	49	30	29	21	16	4	1	3	3

   Using your calculator where necessary, write down the following parameters of the distribution of the number of letters per word, giving your answers accurate to one decimal place, (*e.g.* don't write 12, or 12.28, but 12.3):

   total number of words =
   mode =
   median =
   mean length of words =
   standard deviation =

3. 50 components produced by a machine were categorised, very roughly, by length, into four 'classes', and the following distribution was obtained:

length (in mm):	999 – 1001	1001 – 1003	1003 – 1005	1005 – 1007
number of components:	8	20	16	6

   (a) Draw a clear histogram of the frequency distribution, and also the cumulative frequency graph. What is the modal class?

   (b) Using suitable mid-class values, find the mean and the standard deviation of the lengths, both to the nearest 0.1 mm.
   (Show your working clearly; you will receive up to half the marks for this part if you appear to have used the statistical functions on a calculator.)

   You may find the following formulae useful: mean $\quad m = \dfrac{\sum (x_r - A) f_r}{\sum f_r} + A,$

   variance $\quad s^2 = \dfrac{\sum (x_r - A)^2 f_r}{\sum f_r} - (m - A)^2.$

4. The number of words in the 14 lines of a sonnet by Michael Drayton are 11, 10, 10, 8, 8, 8, 9, 8, 9, 7, 9, 7, 10 and 9 respectively.

(a) Set up a frequency distribution table for the number of words in a line, and draw a histogram and a cumulative frequency diagram.

(b) Find the mode and the median of the number of words per line. Then find the mean, without using the statistical function on the calculator.

(c) Write down the range and the inter-quartile range of the number of words per line. Use your calculator to find the standard deviation.

5. A machine fills bags with sugar, and each bag is supposed to contain 1 kg of sugar. When 50 bags were weighed carefully, the following distribution of weights was found:

weight (in g)	number of bags
998 - 1000	4
1000 - 1002	9
1002 - 1004	18
1004 - 1006	12
1006 - 1008	7

(a) Draw a histogram and a cumulative frequency diagram of the distribution. Use one of your two diagrams to estimate the median weight of sugar in a bag, accurate to one tenth of a gram.

(b) Find the mean and standard deviation of the weights of the 50 bags of sugar, accurate to two decimal places.

6. The arrival times of a group of 60 students at a lecture were recorded, and the following distribution was found; ($x = 0$ is when the lecture starts, and negative times mean that the students arrived early):

arrival time x, in minutes	-6 to -4	-4 to -2	-2 to 0	0 to 2	2 to 4
number of students	8	15	19	12	6

(a) Draw the histogram of this frequency distribution, and

(b) the cumulative frequency curve, in separate diagrams.

(c) On your histogram, show how to locate the median arrival time; (give the value as accurately as your graph allows.)

(d) On your cumulative frequency curve, show how to find the first and third quartile value of the arrival times; (give the values as accurately as your graph allows.)

(e) Using your calculator or the formulae given below, find the mean arrival time, and

(f) the standard deviation of the arrival times.

(Some formulae, if you want to use them: $m = \dfrac{\sum x_r f_r}{\sum f_r}$ $\quad s^2 = \dfrac{\sum x_r^2 f_r}{\sum f_r} - m^2$.)

7. The length of time that the sun was shining on each of the 30 days of June in some year was measured in some place, and the following distribution was found:

length of time (in hours):	0 – 4	4 – 8	8 – 12	12 – 16
number of days:	4	8	12	6

(a) Draw a clear histogram of the distribution, and estimate the mode of the number of hours of sunshine, accurate to the nearest hour.

(b) Calculate the cumulative frequencies, and draw a cumulative frequency diagram.

(c) From one of your two diagrams, estimate the median number of hours of sunshine per day, accurate to the nearest half hour. Show your method clearly.

(d) Using suitable mid-interval values, find the mean number of hours of sunshine, and the standard deviation of the number of hours.

(e) There is a rule which says that, roughly, $mode \approx mean - 3 \cdot (mean - median)$. Substitute the values you estimated in (a) and (c) for the mode and the median into this rule, to find the value of the mean approximately.
What is the percentage error of the approximation in this case?

8. The cumulative frequency distribution of a continuous variable x has the following points on it:

x	35	45	55	65	75	85	95	100
F_r	0	5	15	30	39	45	48	48

Draw a cumulative frequency graph, connecting the given points with a smooth curve. Use your graph to estimate the median of the distribution of the variable.

Copy and complete the following frequency distribution table, using the cumulative frequencies given:

x	x_r	f_r
35 - 45	40	5
45 - 55	50	10

Draw the histogram of this distribution, and without using your calculator, find the mean of this distribution, showing all your working clearly.

9. A variable x is distributed with mean 17 and standard deviation 4.

Another variable, y, is calculated by doubling each value of x and adding 6. What then will be the mean and the standard deviation of y?

10. The variable x is distributed such that its mean is 9 and its standard deviation 2. What are the mean and standard deviation of the variable $y = 3x - 4$?

11. The mean of the three numbers a, $6a$ and x is $3a$. Find x in terms of a.

If the variance of this distribution is 10.5, find the value of a. (Hint: it will be easiest here to use the definition of the variance as the mean square deviation from the mean.)

Normal Distribution

12. A random variable x is normally distributed with mean 1002.8 and standard deviation 1.8 . What are the probabilities that one value of the variable is

(a) greater than 1005,

(b) less than 1001,

(c) greater than 1005 <u>or</u> less than 1001 ?

13. The weights of a random sample of 200 apples from a certain orchard are distributed as follows:

weights (in grams):	100-130	130-160	160-190	190-220	220-250	250-280	280-310
number of apples:	2	7	35	72	60	20	4

Apples with a weight below 180g are only used to make juice; apples whose weight is in the top 25% can be sold at a higher price.

(a) Using suitable mid-class values, find the mean and standard deviation of the distribution of weights.

(b) Find the cumulative frequencies, and draw a smooth cumulative frequency diagram. From your diagram estimate

i. how many apples are only used to make juice, and

ii. the weight above which an apple can be sold at a higher price.

Indicate which points on the diagram you are using to find your answers.

(c) If weights of apples from the orchard are known to be normally distributed, with the mean and standard deviation you found for the sample, find

i. the percentage of all apples which have a weight above 200g,

ii. the weight such that a quarter of all apples weigh less than it.

14. The weights of 40 students in a class at a school (in Britain!) were measured and found to have the following distribution:

weight (in stones):	7 – 8	8 – 9	9 – 10	10 – 11
number students:	2	15	18	5

(a) Draw a clear histogram of the distribution, and a cumulative frequency diagram.

(b) From one of your two diagrams, estimate the median weight. Show your method clearly.

(c) Using suitable mid-interval values, find the mean weight and the standard deviation of the distribution of weights.

(d) Assuming now that the students' weights are normally distributed, with the mean and standard deviation you found in (c), how many students in the class would you expect, on the basis of this model, to have a weight

i. between 8 and 10 stones,

ii. below 9 stones,

iii. above 10 stones 7 pounds? (Note: 1 stone = 14 pounds.)

Compare your answers to i. and ii. to the observed frequencies in the table above. Any comments?

15. The mass of sugar in bags filled by a certain machine is known to be normally distributed with standard deviation 3g. If 95% of the bags are found to have a mass less than 505g what is the mean mass of the bags?

16. A normal distribution has mean $\mu = 120$.

(a) If the standard deviation is $\sigma = 15$, what proportion of values will lie between 100 and 150?

(b) If 20% of the values lie below 100, what is the standard deviation σ ?

17. The accuracy of a machine to fill oil into bottles is such that the standard deviation of the amount of oil is 48 ml. (Write down all your answers accurate to two decimal places.)

(a) If the mean is set to 5080 ml per bottle, then the proportion of bottles that will contain

i. fewer than 5000 ml =

ii. more than 5100 ml =

(b) At the same setting of the machine, if 5% of all bottles are rejected for being too light, then the required amount of oil =

(c) If on another day the machine is set so that only 25% of all bottles contain more than 5100 ml, what is the mean amount of oil per bottle on that day?

18. A machine is used to fill pint-bottles with milk. The precision of the machine is such that the standard deviation of the volume per bottle is 1.2 ml. (You can assume the volume of milk per bottle to be normally distributed.)

(a) At a certain setting of the machine, the mean volume of milk per bottle is 571 ml. What percentage of bottles contain less than 568 ml? In a crate of 20 bottles, how many do you expect will contain less than 568 ml?

(b) One morning, before the machine has been properly set, 55% of all bottles contain less than 568 ml. What is the mean volume of milk per bottle at that setting?

(568 ml, incidentally is one (British) pint; a US pint is smaller.)

19. (a) For some integer k, a sample of values of a variable x was found to have the following frequency distribution:

x_r	4	5	6
f_r	3	$2k$	$k+1$

If the mean of the sample is $m = 5.125$, what is the value of k ?

For that value of k find the standard deviation s of the sample.

(b) The heights of a large group of students is normally distributed with mean $\mu = 169$ cm and standard deviation $\sigma = 12$ cm. What proportion of students do you expect to be taller than 184 cm ?

If three students are selected at random, what is the probability that all three are taller than 184 cm?

(c) Estimate the mean and the standard deviation of the ages of teachers at Atlantic College. (No credit given for flattery!)

20. The distribution in problem 6, above, was for a lecture in the middle of the morning. For another lecture, early in the morning, the mean arrival time was -0.5 minute, and the standard deviation of the arrival times was 2.6 minutes. The students' arrival times can be assumed to have a normal distribution.

(a) Using your calculator or the table, find the proportion of students that will arrive within one minute of the start of the lecture, i.e. between one minute before and one minute after it starts.

(b) Also find the proportion of students that will have arrived on time, i.e. at $x = 0$.

(c) What proportion of students will arrive more than 4 minutes after the lecture has started? If 60 students attend the lecture, how many would you expect to be that late?

(d) After the teacher has taken strong measures, students are more careful. If the mean arrival time is now -1.1 minutes, but 75% of all students are on time, (i.e. they have arrived at $x = 0$,) what is the new standard deviation of the arrival times?

Answers to Selected Problems

2 Algebra, Exponential and Logarithm Functions

p. 6	Qu. 4	(a) i. 5 ii. -2 iii. 3 (b) 1 (c) $2\log a + \log b - 3\log c$ (d) i. 4 ii. 3.97 (e) 1.5

3 Straight Lines, Linear Equations

p. 8	Qu. 2	$M(1,3)$, $y = -\frac{3}{2}x + \frac{9}{2}$, $AB = 2\sqrt{13}$
p. 8	Qu. 3	$K(1,-2), L(4,1), M(0.5,3), N(-2.5,0)$, $m = 1, d = 3\sqrt{2}$: parallelogram
p. 8	Qu. 5	(a) $AB = 3\sqrt{5}$ (b) $y = -2x + 8$ (c) $y = \dfrac{1}{2}x + \dfrac{5}{4}$
p. 8	Qu. 8	(a) $(1,-2)$ (b) $k = -4$ (c) $k = 1$
p. 9	Qu. 9	(a) i. $y = \frac{3}{2}x + \frac{5}{2}$, $2\sqrt{13}$ ii. $(3,0)$, $y = \frac{1}{2}x - \frac{3}{2}$ (b) $x = 0.7, y = -0.6$
p. 9	Qu. 10	one pad costs £ 1.50, one pen £ 0.50
p. 10	Qu. 14	(a) $y = -2x + 1$ (b) $y = \frac{1}{2}x + \frac{7}{2}$ (c) $F(-1,3)$ (d) (e) $3\sqrt{5}$

4 Quadratics, Polynomials, Rational Functions

p. 11	Qu. 1	(a) 0, 6 (b) 3 (c) 1, 2 (d) $-\tfrac{1}{2}$, 2 (e) $\dfrac{-1 \pm \sqrt{13}}{3}$ (f) no such x
p. 11	Qu. 4	$3x^2 + 5x - 2 = 0$ (or any integer multiple of this)
p. 11	Qu. 6	(a) $\alpha + \beta = -\dfrac{3}{2}$, $\alpha\beta = \dfrac{5}{2}$, $5x^2 + 6x + 8 = 0$ (b) $p = 3$
p. 12	Qu. 9	(a) $a = -\dfrac{1}{2}$, $b = 1$, $c = 4$ (b) $V(1, \dfrac{9}{2})$
p. 12	Qu. 12	(a) $y = -2x + 1$ (b) $d = \sqrt{65 - 20t + 5t^2}$ (c) $t = 2 : d = 3\sqrt{5}$
p. 13	Qu. 15	(a) intercepts $(-1,0), (3,0), (0,-6)$, vertex $(1,-8)$ (c) $(2,-6), (-2,10)$ (d) $k = -6$
p. 13	Qu. 18	(a) $(-3,2), (1,6)$ (b) $c = 0$
p. 14	Qu. 19	(a) $(-2,5), (4,17)$ $x = -2.58$, 6.58 (b) $p = -1$
p. 14	Qu. 23	(a) 12 (b) 1 (or 2, or -3), $P(x) = (x-1)(x-2)(x+3)$
p. 15	Qu. 27	(a) $a = -2$, $b = 5$ (b) $Q(x) = 2x + 3$, $R = 1$
p. 15	Qu. 29	(a) $a = \dfrac{2}{3}$, $b = -2$ (b) i. $x = -3$ ii. $x = 2$
p. 15	Qu. 30	$a = -2$, $b = 1$, $c = -1$, $d = \frac{1}{2}$
p. 16	Qu. 31	$y = 1$, $x = -\dfrac{3}{2}$, $0 \le y \le \dfrac{3}{4}$ (can estimate these from the graph)

6 Sequences, Series

p. 18 Qu. 2 $x = 2$, first 4 terms: $4, 5, 6, 7$, or $x = 3$, first 4 terms: $5, 8, 11, 14$

p. 18 Qu. 4 (a) $d = 9$, $a_{23} = 195$, $S_{23} = 2208$, (b) $n = 32$

p. 18 Qu. 5 $r = -2$, next two terms: $-12, 24$, $a_{10} = 1536$, $S_{10} = 1023$

p. 18 Qu. 8 (a) $r = -\frac{4}{3}$ $S_3 = 13$ S_∞ does not exist
(b) $r = \frac{2}{3}$ $S_3 = 19$ $S_\infty = 27$

p. 19 Qu. 13 (a) $r = 2, d = 5$ or $r = 1, d = 0$
(b) i. £ 2120, 2240, 2360, $2000 + 120n$, 3200; after 16.7 years
ii. £ 2090, 2184.05, 2282.33, $2000 \cdot 1.045^n$, 3105.94; end of 16 years
iii. 4.81 %

7 Trigonometry, Circles

p. 21 Qu. 3 (a) 4.76° (b) 4.78°, steeper 598 m

p. 21 Qu. 4 (a) 210°, 330° (b) 30°, 330° (c) 76.0°, 256°

p. 22 Qu. 11 intersections: $(0, 4.60), (5.65, 0)$, $y_{\max} = 5$, when $x = \frac{\pi}{3}$

p. 22 Qu. 12 (a) i. intersections $(\frac{\pi}{3}, 0), (0, -1)$ ii. $\frac{2\pi}{3}, 2\pi$ (b) using calculator: 0.624
(c) greatest = 4.5, least = −1.5

p. 22 Qu. 14 (a) asymptotes $x = \frac{\pi}{2}, \frac{3\pi}{2}$
(b) period = π, amplitude = 3, $y = 3\sin(2x + \frac{\pi}{4}) + 3$

p. 23 Qu. 16 angle = 35.4° area = 31.3 km^2

p. 23 Qu. 19 $C_1 = 41.8°, A_1 = 108.2°$ $C_2 = 138.2°, A_2 = 11.8°$

p. 24 Qu. 22 (b) 0.933 km (c) 0.599 km

p. 24 Qu. 23 (a) 52.2° or 7.78°, 26.5 or 4.53 km, 21.9 km (b) $\frac{31 \pm \sqrt{481}}{2}$, difference = $\sqrt{481}$ (c) for $SM = 26.5$ km, area = 355 km^2, new road = 26.8 km

p. 24 Qu. 24 (a) 5.81 km (b) 328° (c) $14\sqrt{2} = 19.8$ km^2 (d) $\frac{7\sqrt{2}}{2} = 4.95$ km

p. 25 Qu. 26 (a) i. $\frac{20\pi}{3} = 20.9$ cm ii. $(\frac{100\pi}{3} - 25\sqrt{3}) = 61.4$ cm^2 (b) 1.70, 217 cm^2

p. 25 Qu. 28 $\angle AOB = 120°$, area = $48\pi - 36\sqrt{3} = 88.4$ cm^2

p. 25 Qu. 29 59°

p. 25 Qu. 30 (a) 45.6 cm^2 (b) 50 cm

p. 25 Qu. 31 353 cm^2, 700 cm^2, 50.5 %

p. 26 Qu. 32 using (c): $\frac{1}{4}(\sqrt{6} + \sqrt{2})$

p. 26 Qu. 33 (a) $\frac{\pi}{6}, \frac{5\pi}{6}$ (b) 5°, 125°, 185°, 305° (c) $-\frac{\pi}{3} + 2n\pi$

p. 26 Qu. 35 $\pm\frac{2\pi}{3} + 2n\pi$

p. 26 Qu. 36 $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

p. 27 Qu. 39 (a) $\frac{\pi}{2}, \frac{3\pi}{2}$ (b) $140°, 260°$ (c) $45°, 135°, 225°, 315°$

p. 27 Qu. 45 greatest value = 5, smallest positive $x = 0.927$

p. 27 Qu. 46 (a) $\sqrt{2}\sin(x + 45°)$, $x = 0°, 90°, 360°$ (b) $\sqrt{13}$, $-\sqrt{13}$

p. 28 Qu. 49 (a) $x = \pm\frac{\pi}{3} + 2n\pi,\ \pi + 2n\pi$ (b) $\theta = 0,\ \frac{\pi}{5},\ \frac{3\pi}{5},\ \pi$

p. 28 Qu. 52 (a) $\frac{1}{2}(\cos(A - B) - \cos(A + B))$, $2\sin\frac{P+Q}{2}\sin\frac{Q-P}{2}$, $2\sin 5x\sin 2x$
(b) let $\theta = 1$: only iii. is not a contradiction

p. 29 Qu. 56 (a) $\frac{3}{\sqrt{8}}$ (b) $\frac{3}{5}$ (c) $\frac{1}{2}\sqrt{3}$

8 Differentiation

p. 31 Qu. 2 (a) $y' = 3x^2 + 4x - 3$ (b) $y' = 2x - \frac{10}{x^3}$
(c) $y' = 9\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{1}{\sqrt{x}^3}$ (d) $\frac{\mathrm{d}V}{\mathrm{d}x} = 3kx^2$

p. 32 Qu. 6
(a) $14e^{2x-1}$ (b) $4x(x^2 - 3)$
(c) 3 (d) $\frac{9 - x^2}{x^4}$
(e) $\frac{12x^2 - 7x^3}{2\sqrt{2 - x}}$ (f) $2\cos 2x\cos 3x - 3\sin 2x\sin 3x$
(g) $2\sec^2(x - \frac{\pi}{3})$ (h) $\frac{7}{(x+3)^2}$
(i) $\frac{6}{3x - 1}$ (j) $xe^{2x}(2(1 + x)\cos x - x\sin x)$
(k) $\dot{s}(t) = -gt + 10$

p. 32 Qu. 7 (a) $\tan x$ (b) $3x^2(\sin 3x + x\cos 3x)$ (c) 1 (d) $\frac{2(x-2)^2(7-2x)}{x^8}$

p. 32 Qu. 8 (a) $2\ \mathrm{cosec}\ 2x$ (b) $e^{x^2}(1 + 2x^2)$ (c) $\frac{x+2}{x\sqrt{x}}$ (d) $\frac{(x+3)^6(5x-6)}{x^3}$

p. 33 Qu. 9 tangent: $y = -2x + 9$, normal: $y = \frac{1}{2}x + 4$

p. 33 Qu. 10 (a) i. $-2e^{3-2x}$, ii. $e^x(\sin 2x + 2\cos 2x)$, iii. $\frac{1 - \ln x}{x^2}$, iv. $-\tan x$
(b) tangent: $y = 2x - 2$

p. 33 Qu. 12 $x + 3y = 4$ $x = -\frac{8}{3}$

p. 33 Qu. 14 stationary point $(2, 12)$, minimum; $x = -\sqrt[3]{16}$

p. 33 Qu. 15 min at $(-2, -27)$, pt of infl at $(1, 0)$, intersections $(1, 0), (-3, 0), (0, -3)$

p. 34 Qu. 16 (a) min at $(0, 0)$, max at $(-2, 4/e^2)$
(b) as $x \to \infty, y \to \infty$, quickly, and as $x \to -\infty, y \to 0$, intersection $(0, 0)$

p. 34 Qu. 17 (a) $(0, 0)$ (b) max at $(-1, 1/e^2)$, min at $(0, 0)$ (c) $x = -1 \pm \frac{1}{2}\sqrt{2}$

p. 34 Qu. 18 $n = 2$ $m = 2$ greatest value = 22 least value = −21

p. 35 Qu. 22 2048

p. 35 Qu. 23 (a) $y = \frac{2}{9}\sqrt{3} + 4$ (b) 10

p. 36 Qu. 28 (a) $\tan x + \tan y = a$ (c) $t = \dfrac{\sec x}{u} + \dfrac{\sec y}{v}$

(d) $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \dfrac{\sec x \tan x}{u} + \dfrac{\sec y \tan y}{v}\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}t}{\mathrm{d}x} = 0$

p. 37 Qu. 29 (a) $x_0 = 3.3$ (b) $x_1 = 3.294366197$ (c) error $= 1.36 \times 10^{-4}$ %

p. 37 Qu. 31 (a) $\sqrt[3]{7.76} \approx 1.98$, error $= 0.0103$ % (b) $3p$ %, $\frac{1}{3}p$ %

p. 37 Qu. 32 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y - 10x}{2y - x}$, normal $3x - 8y + 13 = 0$

p. 38 Qu. 35 (a) $k = 3 - \pi$, $2xy + (x^2 + 2)y' + \cos y\ y' - 4e^x = 0$, $y = 2x + \dfrac{\pi}{2}$

(b) $4y^3y' + y' = 6x^5 + 4x^3$, 2, $12y^2(y')^2 + 4y^3y'' + y'' = 30x^4 + 12x^2$, $-\dfrac{6}{5}$

p. 38 Qu. 36 $\pi^2 - e^2$, $\dfrac{2\pi + 1}{2e^2}$

p. 38 Qu. 37 $\mathrm{d}y/\mathrm{d}x = -2\cos t, (-\frac{1}{2}, \frac{3}{4})$

p. 38 Qu. 41 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos t}{\cos 2t}$ $(0,0), 1;\ (0,2), 0;\ (0,0), -1;\ (0,-2), 0;\ (0,0), 1$

p. 39 Qu. 42 $A(\dfrac{3}{2}\sqrt{2}, -\sqrt{2})$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2\sin 3\theta}{\cos\theta}$

(a) A function cannot have two y-values for one x-value. (b) Still different y-values for the same x-value, but they happen for different values of θ.

p. 39 Qu. 43 $t = 0, \pi$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3\cos 2t}{\cos t}$, $m = \pm 3$

p. 39 Qu. 44 $\mathrm{d}V/\mathrm{d}t = 200\pi = 628$ cm^3/min, when $r = 9.84$ cm, $\mathrm{d}V/\mathrm{d}t = 609$ cm^3/min

p. 39 Qu. 45 area $= 8\sin\theta$ m^2, 0.14 m^2/day $= 58.3$ cm^2/hr

p. 39 Qu. 46 $\dfrac{20\pi}{9}$ m/sec

9 Integration

p. 40 Qu. 3 (a) i. $2e^x + 3\ln x + c$, ii. $5x - 4\sqrt{x} + c$ (b) 2, 0.2

p. 40 Qu. 4 (a) i. $2x^3 - \frac{2}{3}\sqrt{x}^3 + c$ ii. $\frac{2}{\pi}\sin \pi x + 2e^{2x} + c$ (b) i. $\frac{16}{3}$ ii. $1 - \dfrac{1}{e^2}$

p. 40 Qu. 5 (a) i. $y' = \dfrac{3}{x}$, ii. $y' = \dfrac{5}{x}$, (b) $2\ln\dfrac{5}{3}$

p. 40 Qu. 6 (a) $e^{1+x^2}(1 + 2x^2)$ (b) $\frac{1}{2}\ e^{1+x^2} + c$

p. 40 Qu. 7 $(x^2 + 3)\sin x + 2x\cos x + c$

p. 41 Qu. 9 (a) $-\frac{1}{6}\cos^6 x + c$ (b) $-x\cos 2x + \frac{1}{2}\sin 2x + c$

(c) $\frac{2}{3}\sqrt{x}(x - 3) + c$ (d) $e^{\tan\theta} + c$

(e) $6\arctan\dfrac{x}{3} + c$ (f) $3\ln\left|\dfrac{x-3}{x+3}\right| + c$

(g) $9\ln(x^2 + 9) + c$ (h) $\ln 3$

p. 41 Qu. 10 (a) $\frac{1}{2}x^4 - x^3 + c$ (b) $\frac{2}{3}\arcsin 3x + c$
(c) $\frac{1}{4}\, e^{2x-3}\,(2x-1) + c$ (d) $\ln\sin x + c$
(e) $\frac{1}{4}\sin 2x + \frac{1}{2}x + c$

p. 41 Qu. 12 (a) $\arcsin x + c$ (b) $-\arccos x + c$ $\Leftarrow$ the two functions differ by a constant

p. 42 Qu. 13 $f(x) = \frac{1}{2}e^{2x} + x + 1 - \frac{1}{2}e$

p. 42 Qu. 14 (a) 4 (b) $\ln 3$

p. 42 Qu. 15 (a) $\frac{1}{4}\,x^2\,(2\ln x - 1) + c$ (b) $p = \frac{1}{2},\ q = -1$

p. 42 Qu. 16 $C_1 = 1,\quad \dfrac{16}{35}$

p. 42 Qu. 17 (a) $v = -10t + 6$ (b) $s = -5t^2 + 6t + 8$ (c) $t = 2$ sec, $v = -14$ m/sec

p. 42 Qu. 18 (a) $v(1) = 0$ km/min, $v(4) = 2$ km/min (b) $a(t) = \frac{1}{9}(-4t^2 + 20t - 16)$
(c) $a_{\text{max}} = 1$ km/min^2 at $t = 2.5$ min (d) 6 km (e) 3 km
(f) 9 km, 1.29 km/min

p. 43 Qu. 20 2π (Note: this can be shown from the graph, by inspection.)

p. 43 Qu. 21 intersections $(-2, 6), (3, 11)$, area $= \dfrac{125}{6}$

p. 43 Qu. 23 $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$, area $= \sqrt{3} - \dfrac{\pi}{3}$

p. 43 Qu. 24 $\dfrac{28}{3},\quad \dfrac{32}{3}$

p. 43 Qu. 25 (a) 2, finite area, but infintely high!
(b) i. intersections $(0, 0), (2, 0)$, asymptotes $x = -1,\ 3,\ y = 1$
ii. $\dfrac{6(1-x)}{(x^2 - 2x - 3)^2}$, $(1, 0.25)$ iii. $x + \dfrac{3}{4}\ln\left|\dfrac{x-3}{x+1}\right| + c$ iv. 0.352

p. 44 Qu. 28 (a) $P(\frac{1}{4}\pi, \frac{1}{2}\sqrt{2}),\ Q(-\frac{3}{4}\pi, -\frac{1}{2}\sqrt{2})$ (b) $2\sqrt{2}$ (c) $y = 0.707x + 0.152$
(d) $35.3°,\ 109.5°$ (or $70.5°$)

p. 44 Qu. 29 $\dfrac{1}{2} - \dfrac{1}{k}$, area $\to \dfrac{1}{2}$

p. 44 Qu. 30 (a) $(0, 1)$ (b) maximum, $(\pm 1, 1/\sqrt{e})$ (c) asymptote $y = 0$
(d) $L \approx 2.507,\ k = 2$ (e) $-e^{-x^2/2} + c,\ M = 0$ (f) $V \approx 2.507,\ \dfrac{V}{L} = 1$
(g) normal distribution

p. 46 Qu. 34 $y = C \cdot \dfrac{x-3}{x-2}$, particular integral: $C = 6$

p. 46 Qu. 35 (a) $2\sqrt{3} - \dfrac{2}{3}$ (b) 2.793 (c) $\sqrt{3} \approx 1.730$, error $= -0.126\%$

10 Vectors

p. 47 Qu. 1 $2\sqrt{2} = 2.83$ km/h, SW

p. 47 Qu. 2 $v = 436$ km/h bearing $= 187°$

p. 47 Qu. 3 $d = 13.2$ km bearing $= 296°$

p. 48 Qu. 8 (a) $k = 1, l = 2, m = 3$ (b) $\theta = 45°$

p. 47 Qu. 5 (a) $\vec{x} = \begin{pmatrix} -6.5 \\ -4.5 \end{pmatrix}$ (b) i. $k = 6$ ii. $k = -\frac{3}{2}$

p. 47 Qu. 6 (a) i. $\lambda = -2$ ii. $\lambda = 2$ (b) $\widehat{\vec{x}} = \pm \begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix}$

p. 48 Qu. 7 (a) $-\dfrac{1}{\sqrt{5}}$, about 117° (b) $\dfrac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

p. 48 Qu. 9 (a) $k = -8$ (b) $k = \frac{3}{2}$, $l = -3$ (c) $\theta = 132°$

p. 48 Qu. 10 (a) $|\,\vec{r} + \vec{s}\,| = 5$ (b) $\theta = 94.4°$

p. 48 Qu. 12 (a) $M(2, 3, -1)$ (b) $BC = 6, AM = \sqrt{26}$ (c) $\overrightarrow{BC} \cdot \overrightarrow{AM} = 0$, isosceles triangle (d) area $= 3\sqrt{26}$

p. 49 Qu. 14 (a) $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, 5, $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, 3, $-\dfrac{2}{15}$, area = 7.43 (b) $\begin{pmatrix} 4 \\ 14 \\ -3 \end{pmatrix}$

p. 49 Qu. 15 $\overrightarrow{AC} \cdot \overrightarrow{BC} = (\vec{c} - \vec{a})(\vec{c} + \vec{a}) = \vec{c}^{\,2} - \vec{a}^{\,2} = r^2 - r^2 = 0$

p. 49 Qu. 16 (a) $OB = \sqrt{41}$ (b) $\overrightarrow{BX} = \lambda \begin{pmatrix} 9 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix}$
(c) $\lambda = \frac{22}{39}$, X lies 5.08 km E and 3.38 km N of O

p. 50 Qu. 19 (a) $\vec{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}$ (b) $3x + y + 2z = 3$ (c) 55.2°

p. 50 Qu. 21 (a) not parallel ii. 101° $\vec{r} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, or some such
(b) $\vec{d} \cdot \vec{n} = 0$, so parallel i. distance = 14

11 Complex Numbers

p. 52 Qu. 7 (a) $-\frac{7}{5} - \frac{4}{5}\,i$ (b) $z = \pm(4 + i)$ (c) $z = \pm(1.27 - 0.79\,i)$
(d) $z = 2$ or $z = 1 \pm \sqrt{3}\,i$

p. 53 Qu. 9 $r = e^{e^2/2}$, $\theta = e^2\sqrt{3}/2$

p. 53 Qu. 13 (a) i. circle, centre at $-2 + i$, radius 2 ii. halfline from $-2 + 3\,i$, to the SE
iii. perpendicular bisector of -1 and $-3 + 2\,i$
(b) $3x^2 + y^2 = 3$

12 Linear Transformations and Matrices

p. 55 Qu. 8 (a) i. rotation through $\arctan \dfrac{2}{3} = 33.7°$ and enlargement by $\sqrt{13}$
ii. $P^{-1} = \dfrac{1}{13}\begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$ iii. $PQ = \begin{pmatrix} -11 & -10 \\ 10 & -11 \end{pmatrix}$
(b) i. $z = \sqrt{13}\,(\cos 33.7° + i \sin 33.7°)$
ii. $\dfrac{1}{z} = \dfrac{1}{13}\,(3 - 2i)$ iii. $zw = -11 + 10i$
(c) i. $-\dfrac{26}{25} + \dfrac{7}{25}\,i$ ii. $\dfrac{1}{25}\begin{pmatrix} -26 & -7 \\ 7 & -26 \end{pmatrix}$
reasoning: $z = a + bi$ apparently always corresponds to $P = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

p. 56 Qu. 11 (a) $x = -0.7,\ y = 6.3,\ z = -4.0$ (b) $\lambda = -5$, e.g. $x = 1,\ y = 1,\ z = -3$

p. 57 Qu. 14 (a) $Q = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 4 \end{pmatrix}$

(b) (i) $MF,\ MQ,\ M+F,\ (M+F)Q,$ (ii) $MQ,\ M+F,\ (M+F)Q,$

MQ : numbers of (human and pet) male body parts in each apartment block,

$M+F$: numbers of human and of pet residents in each block,

$(M+F)Q$: numbers of (human and pet) body parts in each block

13 Mathematical Induction

p. 58 Qu. 1 (b) is true. Step:

$(n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + 3(n^2 + n + 1)$.

p. 58 Qu. 2 (b) is true. Step:

$\sum_{r=0}^{n+1} 2^r = \sum_{r=0}^{n} 2^r + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{(n+1)+1} - 1$.

14 Probability

p. 59 Qu. 3 (a) 5/8, (b) 3/8, (c) 1/2, (d) 1/2, (e) 0

p. 59 Qu. 4 (a) 1/6, (b) 1/36, (c) 5/6, (d) 1/12, (e) 35/36

p. 59 Qu. 5 (a) 15/16 (b) 1/60

p. 59 Qu. 6 1/3 (Of the equally likely outcomes, it can't be TT, leaving TH, HT, HH.)

p. 60 Qu. 7 (a) 1/1296 (b) 5/54

p. 60 Qu. 8 (a) 10000 (b) 630 (c) i. 120, ii. $\frac{2}{5}$, iii. 60 sec

p. 61 Qu. 14 (a) $\frac{5}{6}$ (b) $\frac{9}{10}$ (c) $\frac{1}{6}$

p. 61 Qu. 15 (a) 3/5, 13/15 (b) 11/15 mutually exclusive: $p(X \wedge Y) = 0$

p. 61 Qu. 17 $\frac{15}{28}$, independent

p. 61 Qu. 18 2/3, independent, because $p(M) \cdot p(N) = p(M \wedge N)$

p. 62 Qu. 20 (a) 1 (b) 3/8 (c) 1/2 (d) 3/4 (e) 9/20

p. 63 Qu. 24 (b) 0.0018 (c) 0.1016 (d) 9/508

p. 63 Qu. 25 20.1% 0.399%

p. 63 Qu. 26 (a) $\frac{27}{64}$ (b) $\frac{37}{64}$ 11 students

p. 63 Qu. 27 (a) 672 (b) $32 - 240x + 720x^2 - ...$, $x = 0.01$: 29.7, error = 0.00361%

(c) 1.75, $p(1) > p(2)$

p. 64 Qu. 29 (a) 15/1024 (b) 63/64

p. 64 Qu. 31 £ 0.25

p. 64 Qu. 32 $k = \frac{3}{32}$, $\mu = 7$ days $T = 6.55$ days

15 Statistics

p. 65	Qu. 2	$\Sigma f_r = 769$, mode = 2.2, median = 3.9, $m = 5.1$, $s = 3.0$
p. 65	Qu. 4	(b) mode = 8.5, median = 8.75, m = 8.79 (c) range = 4, interquartile range = 2, s = 1.15
p. 66	Qu. 5	(a) median = 1003.3 g (b) $m = 1003.36$ g, $s = 2.25$ g
p. 66	Qu. 6	(c) median ≈ 1.3 (d) $Q_1 \approx -3$, $Q_3 \approx 0.5$ (e) $m = -1.23$ (f) $s = 2.34$
p. 67	Qu. 7	(a) mode = 10, (c) median = 9, (d) $m = 8.67, s = 3.77$, (e) $m \approx 8.5$, error = 1.92%
p. 67	Qu. 8	median ≈ 62, $m = 62.1$
p. 67	Qu. 9	$m_y = 40$, $s_y = 8$
p. 68	Qu. 14	(b) median ≈ 9.2 (c) $m = 9.15, s = 0.760$ (d) i. 32.1 ii. 16.9 iii. 1.51 the fit is very close, perhaps too close
p. 69	Qu. 16	(a) 88.6% (b) 23.8
p. 69	Qu. 17	(a) 4.78%, 33.85% (b) 5001.05 ml (c) 5067.62 ml
p. 70	Qu. 20	(a) 0.294 (b) 0.576 (c) 0.0417, 2.50 (d) 1.63